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On modified extension graphs of a fixed atypicality

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ABSTRACT

In this paper we study extensions between finite-dimensional simple modules over classical Lie superalgebras $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(M|2n)$ and \mathfrak{q}_m . We consider a simplified version of the extension graph which is produced from the Ext^1 -graph by identifying representations obtained by parity change and removal of the loops. We give a necessary condition for a pair of vertices to be connected and show that this condition is sufficient in most of the cases. This condition implies that the image of a finite-dimensional simple module under the Duflo-Serganova functor has indecomposable isotypical components. This yields semisimplicity of Duflo-Serganova functor for $\mathcal{F}in(\mathfrak{gl}(m|n))$ and for $\mathcal{F}in(\mathfrak{osp}(M|2n))$.

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0. Introduction

Let \mathcal{C} be a category of representations of a Lie superalgebra \mathfrak{g} and $\text{Irr}(\mathcal{C})$ be the set of isomorphism classes of simple modules in \mathcal{C} . Assume that the modules in \mathcal{C} are of finite length.¹ In many examples the extension graph of \mathcal{C} is bipartite, i.e. there exists a map $\text{dex} : \text{Irr}(\mathcal{C}) \rightarrow \mathbb{Z}_2$ such that

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¹ This can be replaced by existence of local composition series constructed in [5].

$$(\text{Dex1}) \quad \text{Ext}_{\mathcal{C}}^1(L_1, L_2) = 0 \text{ if } \text{dex}(L_1) = \text{dex}(L_2).$$

In what follows $\mathcal{F}in(\mathfrak{g})$ stands for the full subcategory of the category of finite-dimensional \mathfrak{g} -modules consisting of the modules which are completely reducible over $\mathfrak{g}_{\bar{0}}$. In this paper we consider the following examples:

(KM) $\mathcal{C} = \mathcal{F}in(\mathfrak{g})$ with $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(M|2n), D(2|1, a), G(3)$ or $F(4)$.

$(\mathfrak{q}; \frac{1}{2})$ $\mathcal{C} = \mathcal{F}in(\mathfrak{q}_m)_{1/2}$ which is the full subcategory of $\mathcal{F}in(\mathfrak{q}_m)$ consisting of the modules with “half-integral” weights;

$(\mathfrak{q}; \mathcal{C})$ \mathcal{C} is a subcategory of $\mathcal{F}in(\mathfrak{q}_m)$ described in 4.7.

Take \mathfrak{g} as in the (KM)-case. Let $\mathcal{C}(\mathfrak{g})$ be the full subcategory of $\mathcal{F}in(\mathfrak{g})$ consisting of modules with all subquotients having the maximal atypicality (equal to the defect of \mathfrak{g}). The Kac-Wakimoto conjecture, which was formulated in [25] and proven in [32], states that for a simple finite-dimensional module L one has $\text{sdim } L \neq 0$ if and only if L is of maximal atypicality (i.e., $L \in \text{Irr}(\mathcal{C}(\mathfrak{g}))$). It turns out that (see [24], [15], [17] for the proofs) that there are exactly two possible maps $\text{dex} : \text{Irr}(\mathcal{C}(\mathfrak{g})) \rightarrow \mathbb{Z}_2$ satisfying (Dex1): the map given by $\text{dex}(L) := \bar{0}$ if and only if $\text{sdim } L > 0$ and the map $\text{dex}' := 1 - \text{dex}$. Note that the map dex is compatible with the Duflo-Serganova functors introduced in [7] that is

$$(\text{Dex2}) \quad \text{one has } [\text{DS}_x(L) : L'] = 0 \text{ if } \text{dex}(L) \neq \text{dex}(L'),$$

since $\text{sdim } N = \text{sdim } \text{DS}_x(N)$ for any \mathfrak{g} -module N . Note that in (Dex2) we have to choose dex on $\text{Irr}(\mathcal{C})$ and on $\text{Irr}(\text{DS}_x(\mathcal{C}))$.

By [24], [15] in the (KM)-case there exists dex satisfying (Dex1) and (Dex2). Another example when (Dex1) and (Dex2) hold is the full subcategory of integrable modules in the category $\mathcal{O}(\mathfrak{gl}(1|n)^{(1)})$, see [18]. Note that $\mathcal{F}in(\mathfrak{g})$ coincides with the full subcategory of integrable modules in the category $\mathcal{O}(\mathfrak{g})$ if $\dim \mathfrak{g} < \infty$. This suggests the following conjecture: if \mathfrak{g} is a Kac-Moody superalgebra, then the full subcategory of integrable modules in the category $\mathcal{O}(\mathfrak{g})$ admits a map satisfying (Dex1) and (Dex2).

The original motivation for this project was to study complete reducibility of $\text{DS}_x(L)$, where L is a finite-dimensional simple \mathfrak{g} -module. Clearly, the existence of a map satisfying (Dex1) and (Dex2) implies complete reducibility of $\text{DS}_x(L)$ for each $L \in \text{Irr}(\mathcal{C})$. In particular, $\text{DS}_x(L)$ is completely reducible for each simple finite-dimensional module L over a finite-dimensional Kac-Moody superalgebra. Unexpectedly, it turns out that the complete reducibility holds in the cases $(\mathfrak{q}; \frac{1}{2})$ and $(\mathfrak{q}; \mathcal{C})$ even though (Dex2) does not hold. Below we will explain how the complete reducibility can be obtained in the absence of the property (Dex2).

Our main result is formulated in terms of “arc/arch diagrams” used in [24], [10], [22] and in [17]; for \mathfrak{q}_n -case a modification of these diagrams is used in [20].

Theorem A. *Let $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(M|2n)$ or \mathfrak{q}_m . If there exists a non-split extension between two non-isomorphic finite-dimensional simple modules, then the weight diagram of one of these modules can be obtained from the weight diagram of the other module by moving one or two symbols \times along one of the arches. For $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(M|2n)$ and for half-integral weights of \mathfrak{q}_m this condition is sufficient.*

The above condition is not sufficient for integral weights of \mathfrak{q}_m , but it is sufficient if the modules are “large enough”, see Corollary 4.6.5.

Theorem A implies that dex , introduced by the formula (23) below, satisfies (Dex1) in all cases we consider. The computations of $\text{DS}_x(L)$ in [24], [15], [22] imply that the arch diagram corresponding to a subquotient of $\text{DS}_x(L)$ can be obtained from the arch diagram of L by sequential removal of several maximal arches. This, together with Theorem A, implies that any extension between non-isomorphic simple subquotients of $\text{DS}_x(L)$ splits. This gives the complete reducibility of $\text{DS}_x(L)$ in the cases (KM), $(\mathfrak{q}; \frac{1}{2})$, $(\mathfrak{q}; \mathcal{C})$ and the fact that in the \mathfrak{q}_m -case any indecomposable submodule of $\text{DS}_x(L)$ is “isotypical” in the sense of 1.2.3.

The property (Dex1) means that dex gives a bipartition of a certain simplified version of Ext^1 -graph. This graph can be described in the following way. Denote by Π the parity change functor and by $L(\lambda)$ a simple \mathfrak{g} -module of the highest weight λ ; we set

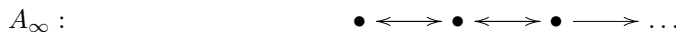
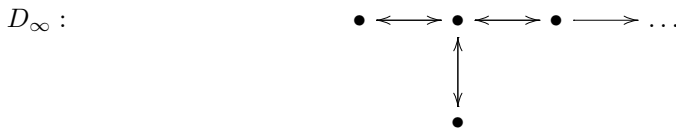
$$\text{ext}_{\mathfrak{g}}(\lambda; \nu) = \begin{cases} \dim \text{Ext}_{\mathcal{O}}^1(L(\lambda), L(\nu)) & \text{if } L(\nu) \cong \Pi L(\lambda) \\ \dim \text{Ext}_{\mathcal{O}}^1(L(\lambda), L(\nu)) + \dim \text{Ext}_{\mathcal{O}}^1(L(\lambda), \Pi L(\nu)) & \text{otherwise.} \end{cases} \quad (1)$$

Let $(\mathcal{C}; \text{ext})$ be the graph with the set of vertices $\text{Irr}(\mathcal{C})$ modulo the involution defined by Π , with $L(\lambda)$ and $L(\nu)$ connected by $\text{ext}_{\mathfrak{g}}(\lambda; \nu)$ edges if $\lambda \neq \nu$. This graph can be obtained from the usual Ext^1 -graph in two steps: factoring modulo the involution followed by deleting the loops. The graph obtained by factoring modulo the involution does not have loops if \mathfrak{g} is a Kac-Moody superalgebra and $\mathcal{C} \subset \mathcal{O}(\mathfrak{g})$; in the \mathfrak{q}_m -case there is at most one loop around each vertex and the vertices with loops correspond to the weights having at least one zero coordinate, see Theorem 3.1 of [21]. The property (Dex1) means that dex gives a bipartition of the graph $(\mathcal{C}; \text{ext})$.

In many cases Ext^1 -graph is isomorphic to two disjoint copies of $(\mathcal{C}; \text{ext})$ which is equivalent to the following property: the vertices L and ΠL lie in the different connected component of the Ext^1 -graph. It is easy to see that this property holds if \mathfrak{g} is a Kac-Moody superalgebra and $\mathcal{C} \subset \mathcal{O}(\mathfrak{g})$. For $\text{Fin}(\mathfrak{q}_m)_{1/2}$ this property follows from results of [1]. This property does not hold for the atypical integral blocks in $\text{Fin}(\mathfrak{q}_m)$.

0.1. Remark

It was observed by Alex Sherman, our results imply the following: each block of atypicality k^2 in $\mathcal{F}in(\mathfrak{g})$ contains a “large” Serre subcategory \mathcal{C}_+ (described in 4.5.4) such that the graph $(\mathcal{C}_+; \text{ext})$ is isomorphic to $(\mathcal{C}_{1/2}(k); \text{ext})$, where $\mathcal{C}_{1/2}(k)$ is the half-integral block of atypicality k in $\mathcal{F}in(\mathfrak{q}_{2k})$. In order to illustrate this observation, consider the simplest case $k = 1$ which was studied in [12], [13], [26], [27] and [21]. We have the following three types of ext-graphs:



The first graph corresponds to the blocks of atypicality one in $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(2m|2n)$ and some blocks of atypicality one in $F(4)$ and $D(2|1; a)$ for $a \in \mathbb{Q}$; the second graph corresponds to the blocks of atypicality one in $\mathfrak{osp}(2m + 1|2n)$, $\mathfrak{osp}(2m|2n)$, $G(3)$ and the rest of the blocks of atypicality one in $F(4)$ and $D(2|1; a)$. The third graph corresponds to the blocks of atypicality one for \mathfrak{q}_m ; this graph is contained in the first two graphs. The picture is much more complicated for $k > 1$. For instance, the vertices of the blocks of atypicality two in $\mathcal{F}in(\mathfrak{g})$ are enumerated by the integral pairs (i, j) satisfying the following conditions:

- $i < j$ for $\mathfrak{gl}(m|n)$
- $0 < i < j$ or $i = j = 0$ for $\mathfrak{osp}(2m + 1|2n)$, the integral blocks for \mathfrak{q}_m and certain blocks for $\mathfrak{osp}(2m|2n)$;
- $|i| < j$ or $i = j = 0$ for the rest of the atypicality two blocks for $\mathfrak{osp}(2m|2n)$;
- $0 < i < j$ for the half-integral blocks for \mathfrak{q}_m .

The last graph is an induced subgraph³ of all above graphs except, perhaps, for the integral blocks for $\mathfrak{g} = \mathfrak{q}_m$; for the latter case the last graph is isomorphic to the induced subgraph for the vertices (i, j) with $1 < i < j$.

² In contrast to the Kac-Moody case, the definition of the degree of atypicality in the \mathfrak{q}_n -case admits several variations; for example, our degree of atypicality is the integral part of the degree appearing in [20].
³ the induced subgraph is the graph with the set of vertices B which includes all edges $\mu \rightarrow \nu$ for $\mu, \nu \in B$.

0.2. Methods

For the cases (KM), $(\mathfrak{q}; \frac{1}{2})$ the categories $\mathcal{F}in(\mathfrak{g})$ were studied in many papers including [28], [3], [1], [4], [8], [9] and Theorem A can be deduced from the results of these papers. The categories $\mathcal{F}in(\mathfrak{q}_2), \mathcal{F}in(\mathfrak{q}_3)$ were described in [27] and [21] respectively. In this paper we obtain Theorem A using the approach of [28], [21]. The assertion can be reduced to the case when \mathfrak{g} is one of the algebras $\mathfrak{gl}(n|n), \mathfrak{osp}(2n + t|2n), \mathfrak{q}_{2n+\ell}$ with $t = 0, 1, 2$ and $\ell = 0, 1$ and the simple modules have the same central character as the trivial module. We take \mathfrak{g} as above and denote by \mathcal{B} the set of λ s such that $L(\lambda)$ is finite-dimensional and have the same central character as the trivial module. Our main tools are the functors $\Gamma_{\bullet}^{\mathfrak{g}, \mathfrak{p}}$ introduced in [29] (we use the “dual version” that appeared in [22]). For a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ the functor $\Gamma_{\bullet}^{\mathfrak{g}, \mathfrak{p}} : \mathcal{F}in(\mathfrak{p}) \rightarrow \mathcal{F}in(\mathfrak{g})$ is the derived functor of the functor which maps each finite-dimensional \mathfrak{p} -module to the maximal finite dimensional quotient of the induced module $\mathcal{U}(\mathfrak{p}) \otimes_{\mathcal{U}(\mathfrak{q})} V$. We fix a “nice chain” of Lie superalgebras $\mathfrak{g}_{(0)} \subset \mathfrak{g}_{(1)} \subset \dots \subset \mathfrak{g}_{(n)}$ where $\mathfrak{g}_{(i)} = \mathfrak{gl}(i|i)$ for $\mathfrak{g} = \mathfrak{gl}(n|n)$, $\mathfrak{g}_{(i)} = \mathfrak{osp}(2i + t|2i)$, for $\mathfrak{g} = \mathfrak{osp}(2n + t|2k)$ with $t = 0, 1, 2$ and $\mathfrak{g}_{(i)} = \mathfrak{q}_{2i+\ell}$ for $\mathfrak{g} = \mathfrak{q}_{2n+\ell}$ with $\ell = 0, 1$. For each s the algebra $\mathfrak{p}_{(s)} := (\mathfrak{g}_{(s-1)} + \mathfrak{b}) \cap \mathfrak{g}_{(s)}$ is a parabolic subalgebra in $\mathfrak{g}_{(s)}$. For $\mathfrak{p} := \mathfrak{p}_{(n)}$ the multiplicities $K^i(\lambda; \nu) := [\Gamma_i^{\mathfrak{g}, \mathfrak{p}}(L_{\mathfrak{p}}(\lambda)) : L_{\mathfrak{g}}(\nu)]$ were computed in [28], [22], [30], [31]. We will present the corresponding Poincaré polynomials $K^{\lambda, \nu}(z) := \sum_i K^i(\lambda; \nu) z^i$ in terms of the arch diagram. The same Poincaré polynomials appear in the character formulae obtained in [22], [4], [35] and [16] (in particular, the arch diagrams in \mathfrak{q}_m -case are similar to the diagrams appeared in [35], 3.3). The multiplicities

$$K_{(s)}^i(\lambda; \nu) := [\Gamma_i^{\mathfrak{g}_{(s)} + \mathfrak{b}, \mathfrak{p}_{(s)} + \mathfrak{b}}(L_{\mathfrak{p}_{(s)} + \mathfrak{b}}(\lambda)) : L_{\mathfrak{g}}(\nu)]$$

can be easily expressed in terms of $K^i(\lambda; \nu)$ computed for $\mathfrak{g}_{(i)}$. Set $k_0(\lambda; \nu) := \max_s K_{(s)}^0(\lambda; \nu)$. It turns out that $k_0(\lambda; \nu) \neq 0$ implies that the weight diagram of λ can be obtained from the weight diagram of ν by moving one or two symbols \times along one of the arches. For $\mathfrak{g} = \mathfrak{gl}(n|n), \mathfrak{osp}(2n + t|2n)$ the inequality $k_0(\lambda; \nu) \neq 0$ forces $k_0(\lambda; \nu) = 1$ and $\text{dex}(\lambda) \neq \text{dex}(\nu)$ for the grading dex given by the formula (23). For $\mathfrak{g} = \mathfrak{q}_m$ -case the same hold if ν does not have zero coordinates.

It is not hard to show that $\text{ext}_{\mathfrak{g}}(\lambda; \nu) \leq k_0(\lambda; \nu)$ for $\lambda, \nu \in \mathcal{B}$ with $\nu < \lambda$. This gives the first claim of Theorem A for the case when the highest weights of the modules lie in \mathcal{B} . In Corollary 4.5.1 we show that $\text{ext}_{\mathfrak{g}}(\lambda; \nu) = k_0(\lambda; \nu)$ for $\lambda, \nu \in \mathcal{B}$ with $\nu < \lambda$ except for the case $\mathfrak{g} = \mathfrak{q}_m$ and λ has a coordinate which equals to 0 and to $1 + \ell$. This gives the second claim of Theorem A for the case when the highest weights of the modules lie in \mathcal{B} . The proof of the formula $\text{ext}(\lambda; \nu) = k_0(\lambda; \nu)$ is based on the fact that for each s the radical of the maximal finite dimensional quotient of the induced module $\mathcal{U}(\mathfrak{p}) \otimes_{\mathcal{U}(\mathfrak{q})} L_{\mathfrak{p}_{(s)}}(\lambda)$ is semisimple (the chain $\mathfrak{g}_{(0)} \subset \mathfrak{g}_{(1)} \subset \dots \subset \mathfrak{g}_{(n)}$ is chosen so that this property holds).

0.3. Content of the paper

In Section 1 we present some background information about Ext^1 and establish the inequality $\text{ext}_{\mathfrak{g}}(\lambda; \nu) \leq k_0(\lambda; \nu)$. In Corollary 1.9.3 we obtain the formula $\text{ext}_{\mathfrak{g}}(\lambda; \nu) = k_0(\lambda; \nu)$ under certain assumptions.

In Section 2 we introduce the language of “arch diagrams”.

In Section 3 we deduce a description of the Poincaré polynomials $K^{\lambda, \nu}(z)$ in terms of the arch diagram from results of [30], [31], [28] and [22].

In Section 4 we introduce the \mathbb{Z}_2 -grading dex and show that $K^{\lambda, \nu}(z)$ has nice properties with respect to this grading. Then we compute $\text{ext}_{\mathfrak{g}}(\lambda; \nu)$ for $\lambda, \nu \in \mathcal{B}$ under the assumption that for $\mathfrak{g} = \mathfrak{q}_m$ all coordinates of λ differ from 0, 1 and $1 + \ell$. In 4.6 we establish Theorem A by reducing the computations of $\text{ext}_{\mathfrak{g}}(\lambda; \nu)$ for $\lambda, \nu \in P^+(\mathfrak{g})$ to the case $\lambda, \nu \in \mathcal{B}$. In 4.7 we discuss the conditions (Dex1), (Dex2) in various cases. Finally, in Remark 4.8 we discuss the connection between the ext -graph and Ext^1 -graph.

0.3.1.

This paper has a considerable overlap (the cases of \mathfrak{gl} and \mathfrak{osp}) with the unpublished preprint [14] where the (KM) case was studied.

0.4. Acknowledgments

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0.5. Index of definitions and notation

Throughout the paper the ground field is \mathbb{C} ; \mathbb{N} stands for the set of non-negative integers. We denote by Π the parity change functor. We will use the notation $\text{Soc } N$, $\text{Rad } N$, $\text{coSoc } N$ for the socle, the radical and the cosocle of a module N (recall that $\text{Soc } N$ is the sum of simple submodules, $\text{Rad } N$ is the intersection of maximal submodules and $\text{coSoc } N := N/\text{Rad } N$). Throughout the paper \equiv will be always used for **the equivalence modulo 2**.

$\mathfrak{g}, \mathfrak{t}, (\text{Asst})$	1.2
$\mathcal{O}(\mathfrak{g}), \mathcal{F}in(\mathfrak{g}), C_\lambda, M(\lambda), L(\lambda), P^+(\mathfrak{g})$	1.2.2
$[N : L]$	1.2.3
$\mathcal{N}(\lambda; \nu; m), m(\mathfrak{g}, \mathfrak{p}; \lambda; \nu)$	1.3.3
$\Gamma_i^{\mathfrak{g}, \mathfrak{p}}, K^j(\lambda; \nu)$	1.7.1
$\mathfrak{g}_{(s)}, \mathfrak{p}_{(s)}, \mathfrak{h}_{(s)}, \mathfrak{t}_{(s)}, \mathfrak{h}_{(s)}^\perp, \mathfrak{t}_{(s)}^\perp$	1.8
assumptions (A), (B)	1.8.1
$K_{(s)}^i(\lambda; \nu), \text{ext}_{(s)}(\lambda; \nu)$	1.8.2

$s(\lambda; \nu), k(\lambda; \nu),$ graph $G(t^*; K^0), B(\lambda), K^i$ -stable	1.9
triangular decompositions, ρ	2.1
$\mathcal{B}_0, \mathcal{B}_{1/2}, \mathcal{B}, \lambda_i$	2.2
arch diagram	2.3
$\mathfrak{g}_{(i)}$ for $\mathfrak{g} = \mathfrak{gl}(n n), \mathfrak{osp}(2n + t 2n), \mathfrak{q}_{2n+t}$; tail λ	2.5
$(g)_p^q, (g)_{0,0}^{p,q}, K(\frac{t}{g})$	3.1
$\text{dex}(\lambda), \text{dex}(\lambda; \nu), \text{tail}(\lambda; \nu)$	(23)

1. Useful facts about $\text{ext}(\lambda; \nu)$

1.1.

Lemma. *Let A be an associative superalgebra.*

(i) *If N is an A -module with a semisimple radical and a simple cosocle L' , then*

$$\dim \text{Hom}(L, N) \leq \dim \text{Ext}^1(L', L),$$

for any simple A -module $L \not\cong L'$.

(ii) *Let L_1, \dots, L_s, L' be simple non-isomorphic A -modules and m_1, \dots, m_s be non-negative integers satisfying $m_j \leq \dim \text{Ext}^1(L', L_j)$. There exists an A -module N with*

$$\text{coSoc } N \cong L', \quad \text{Rad } N \cong \bigoplus_j L_j^{\oplus m_j}.$$

Proof. Consider any exact sequence of the form

$$0 \rightarrow L^{\oplus m} \xrightarrow{\iota} N' \xrightarrow{\phi} L' \rightarrow 0.$$

For each $i = 1, \dots, m$ let $p_i : L^{\oplus m} \rightarrow L$ be the projection to the i th component and let $\theta_i : L \rightarrow L^{\oplus m}$ be the corresponding embedding $p_i \theta_i = Id_L$. Consider a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & L^{\oplus m} & \xrightarrow{\iota} & N' & \xrightarrow{\phi} & L' \longrightarrow 0 \\
 & & \uparrow \theta_i & & \downarrow \psi_i & & \downarrow Id \\
 & & p_i & & & & \\
 0 & \longrightarrow & L & \xrightarrow{\iota_i} & M^i & \xrightarrow{\phi_i} & L' \longrightarrow 0
 \end{array} \tag{2}$$

where $\psi_i : N' \rightarrow M^i$ is a surjective map with $\text{Ker } \psi_i = \iota(\text{Ker } p_i)$. The bottom line of this diagram is an element of $\text{Ext}^1(L', L)$, which we denote by Φ_i .

Assume that $m > \dim \text{Ext}^1(L', L)$. Then $\{\Phi_i\}_{i=1}^m$ are linearly dependent and we can assume that $\Phi_1 = 0$. This means that Φ_1 splits, so there exists a projection $\tilde{p} : M^1 \rightarrow L$

with $\iota_1 \tilde{p} = Id_L$. Then $\tilde{p} \circ \psi_1 \circ \iota \circ \theta_1 = Id_L$, so $\tilde{p} \circ \psi_1 : N' \rightarrow L$ is surjective. Therefore $[\text{coSoc } N' : L] \neq 0$, that is $\text{coSoc } N' \not\cong L'$.

Now take N as in (i). Let N' be the quotient of N by the sum of all simple submodules which are not isomorphic to L . One has

$$\text{coSoc } N' = \text{coSoc } N \cong L', \quad \text{Rad } N' \cong L^{\oplus m}$$

where $m = \dim \text{Hom}(L, N)$. By above, $m \leq \dim \text{Ext}^1(L', L)$; this gives (i).

For (ii) let $\{\Phi_i^{(j)}\}_{i=1}^{m_j}$ be linearly independent elements in $\text{Ext}^1(L', L_j)$:

$$\Phi_i^{(j)} : \quad 0 \rightarrow L_j \xrightarrow{\iota_i} M^i \xrightarrow{\phi_i} L' \rightarrow 0.$$

Consider the exact sequence

$$0 \rightarrow \bigoplus_j L^{\oplus m_j} \rightarrow \bigoplus_j \bigoplus_i M_i \rightarrow (L')^{\oplus \sum_{j=1}^s m_j} \rightarrow 0.$$

Let $\text{diag}(L')$ be the diagonal copy of L' in $(L')^{\oplus \sum_{j=1}^s m_j}$ and let N be the preimage of $\text{diag}(L')$ in $\bigoplus_j \bigoplus_i M^i$. This gives the exact sequence

$$0 \rightarrow \bigoplus_j L^{\oplus m_j} \xrightarrow{\iota} N \xrightarrow{\phi} L' \rightarrow 0$$

and the commutative diagrams similar to (2). Assume that $N_1 \subsetneq N$ is a submodule of N satisfying $\phi(N_1) \neq 0$. Since $\text{Ker } \phi = \text{Im } \iota$ is completely reducible we have

$$\text{Ker } \phi = (\text{Ker } \phi \cap N_1) \oplus N_2$$

where $N_2 \neq 0$ is completely reducible. Then $N = N_1 \oplus N_2$ and thus N can be written as $N = L \oplus N_3$ where $L \subset N_2$ is simple. We can assume that $L \cong L_1$. Changing the basis in the span of $\{\Phi_i^{(1)}\}_{i=1}^{m_1}$, we can assume that $\iota(\text{Ker } p_1) \subset N_3$. Since $\text{Ker } \psi_1 = \iota(\text{Ker } p_1)$, $\psi_1(L)$ is a non-zero submodule of M_1 , so the exact sequence Φ_1 splits, a contradiction.

Hence for each $N_1 \subsetneq N$ one has $\phi(N_1) = 0$ that is $\text{Rad } N = \text{Ker } \phi$ as required. \square

1.2. Notation and assumptions

Let \mathfrak{g} be a Lie superalgebra of at most countable dimension with a finite-dimensional even subalgebra \mathfrak{t} satisfying

(Asst) \mathfrak{t} acts diagonally on \mathfrak{g} and $\mathfrak{g}_{\mathfrak{t}}^{\mathfrak{t}} = \mathfrak{t}$.

We set $\mathfrak{h} := \mathfrak{h}^{\mathfrak{t}}$ and choose $h \in \mathfrak{t}$ satisfying

$$\mathfrak{h}^h = \mathfrak{h} \quad \text{and each non-zero eigenvalue of } \text{ad } h \text{ has a non-zero real part.} \quad (3)$$

(The assumption on $\dim \mathfrak{g}$ ensures the existence of h). We write $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta(\mathfrak{g})} \mathfrak{g}_{\alpha})$ with $\Delta(\mathfrak{g}) \subset \mathfrak{t}^*$ and

$$\mathfrak{g}_\alpha := \{g \in \mathfrak{g} \mid [h, g] = \alpha(h)g \text{ for all } h \in \mathfrak{t}\}.$$

We introduce the triangular decomposition $\Delta(\mathfrak{g}) = \Delta^+(\mathfrak{g}) \amalg \Delta^-(\mathfrak{g})$, with

$$\Delta^\pm(\mathfrak{g}) := \{\alpha \in \Delta(\mathfrak{g}) \mid \pm \operatorname{Re} \alpha(h) > 0\},$$

and define the partial order on \mathfrak{t}^* by

$$\lambda > \nu \text{ if } \nu - \lambda \in \mathbb{N}\Delta^-.$$

We set $\mathfrak{n}^\pm := \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha$ and consider the Borel subalgebra $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}^+$.

1.2.1.

Take $z \in \mathfrak{t}$ satisfying

$$\alpha(z) \in \mathbb{R}_{\geq 0} \text{ for } \alpha \in \Delta^+ \text{ and } \alpha(z) \in \mathbb{R}_{\leq 0} \text{ for } \alpha \in \Delta^-. \tag{4}$$

Consider the superalgebra $\mathfrak{p}(z) := \mathfrak{g}^z + \mathfrak{b}$. Notice that

$$\mathfrak{p}(z) = \mathfrak{g}^z \ltimes \mathfrak{m}, \text{ where } \mathfrak{m}(z) := \sum_{\alpha \in \Delta: \alpha(z) > 0} \mathfrak{g}_\alpha.$$

Both triples $(\mathfrak{p}(z), \mathfrak{t}, h)$, $(\mathfrak{g}^z, \mathfrak{t}, h)$ satisfy (Asst) and (3). One has $(\mathfrak{g}^z)^\mathfrak{t} = \mathfrak{p}^\mathfrak{t} = \mathfrak{h}$ and

$$\begin{aligned} \Delta^+(\mathfrak{p}(z)) &= \Delta^+(\mathfrak{g}), & \Delta^+(\mathfrak{g}^z) &= \{\alpha \in \Delta^+(\mathfrak{g}) \mid \alpha(z) = 0\} \\ \Delta^-(\mathfrak{p}(z)) &= \Delta^-(\mathfrak{g}^z) = \{\alpha \in \Delta^-(\mathfrak{g}) \mid \alpha(z) = 0\} \end{aligned}$$

1.2.2. Modules $M(\lambda), L(\lambda)$

For a semisimple \mathfrak{t} -module N we denote by N_ν the weight space of the weight ν . We denote by $\mathcal{O}(\mathfrak{g})$ the full subcategory of finitely generated modules with a diagonal action of \mathfrak{t} and locally nilpotent action of \mathfrak{n} . Recall that $\mathcal{F}in(\mathfrak{g})$ is the full subcategory of the category of finite-dimensional \mathfrak{g} -modules consisting of modules which are completely reducible over $\mathfrak{g}_{\bar{0}}$. Since $\mathfrak{g}_{\bar{0}}$ is reductive, complete reducibility of a finite-dimensional $\mathfrak{g}_{\bar{0}}$ -module is equivalent to complete reducibility of \mathfrak{h} -action. Therefore $\mathcal{F}in(\mathfrak{g})$ is the full subcategory of the finite-dimensional \mathfrak{g} -modules in $\mathcal{O}(\mathfrak{g})$.

By Dixmier generalization of Schur’s Lemma (see [6]), up to a parity change, the simple \mathfrak{h} -modules are parametrized by $\lambda \in \mathfrak{t}^*$; we denote by C_λ a simple \mathfrak{h} -module, where \mathfrak{t} acts by λ (for each λ we choose a grading on C_λ). We view C_λ as a \mathfrak{b} -module with the zero action of \mathfrak{n} and set

$$M(\lambda) := \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} C_\lambda.$$

The module $M(\lambda)$ has a unique simple quotient which we denote by $L(\lambda)$. We set

$$P^+(\mathfrak{g}) := \{\lambda \in \mathfrak{t}^* \mid \dim L(\lambda) < \infty\}.$$

We introduce similarly the modules $M_{\mathfrak{p}}(\lambda)$, $L_{\mathfrak{p}}(\lambda)$ for the algebra \mathfrak{p} .

1.2.3.

For $N \in \mathcal{O}(\mathfrak{g})$ we will denote by $[N : L]$ the number of simple quotients isomorphic to L or to ΠL in a Jordan-Hölder series of N . We say that N is an “ L -isotypical” if all these quotients are isomorphic to L or to ΠL . We introduce $\text{ext}_{\mathfrak{g}}(\lambda; \nu)$ by (1); we will usually drop the index \mathfrak{g} and write simply $\text{ext}(\lambda; \nu)$.

1.3. Remarks

By Lemma 1.1 $\text{Ext}_{\mathcal{O}}^1(L(\lambda), L(\nu)) = \text{Ext}^1(L(\lambda), L(\nu))$ if $\lambda \neq \nu$.

1.3.1.

If \mathfrak{g} is a Kac-Moody superalgebra, then $\{C_{\lambda}\}_{\lambda \in \mathfrak{t}^*}$ can be chosen in such a way that $\text{Ext}_{\mathcal{O}}^1(L(\lambda), \Pi L(\nu)) = 0$ for all λ, ν and thus $\text{ext}(\lambda; \nu) = \dim \text{Ext}_{\mathcal{O}}^1(L(\lambda), L(\nu))$.

If $\mathfrak{g} = \mathfrak{q}_n$, then $\text{ext}(\lambda; \nu) \neq 0$ implies that both C_{λ}, C_{ν} are either Π -invariant or not Π -invariant; in this case for $\lambda \neq \nu$ one has $\text{ext}(\lambda; \nu) = \dim \text{Ext}_{\mathcal{O}}^1(L^{\Pi}(\lambda), L^{\Pi}(\nu))$, where $L^{\Pi}(\lambda)$ is the “ Π -invariant simple module” appeared in [30], [31] (in other words, $L^{\Pi}(\lambda)$ is a simple $\mathfrak{q}_1 \times \mathfrak{q}_n$ -module).

1.3.2.

If \mathfrak{g} is a Kac-Moody superalgebra, then \mathfrak{g} admits antiautomorphism which stabilizes the elements of \mathfrak{t} and the category $\mathcal{O}(\mathfrak{g})$ admits a duality functor $\#$ with the property $L^{\#} \cong L$ for each simple module $L \in \mathcal{O}(\mathfrak{g})$. By [11], for the \mathfrak{q}_n -case $\mathcal{O}(\mathfrak{g})$ admits a duality functor $\#$ with the property $L^{\#} \cong L$ up to a parity change. In both cases

$$\text{ext}(\lambda; \nu) = \text{ext}(\nu; \lambda). \tag{5}$$

1.3.3. Set $\mathcal{N}(\lambda; \nu; m)$

Recall that $h \in \mathfrak{t}$ satisfies (3). Let $\lambda \neq \nu \in \mathfrak{t}^*$ be such that $\text{Re}(\lambda - \nu)(h) \geq 0$. If

$$0 \rightarrow L(\nu) \rightarrow E \rightarrow L(\lambda) \rightarrow 0$$

is a non-split exact sequence, then E is generated by $E_{\lambda} \cong C_{\lambda}$, so E is a quotient of $M(\lambda)$ and $\nu < \lambda$. For $\lambda, \nu \in \mathfrak{t}^*$ we denote by $\mathcal{N}(\lambda; \nu; m)$ the set of \mathfrak{g} -modules N satisfying

$$\text{coSoc } N \cong L(\lambda); \quad \text{Soc } N = \text{Rad } N \text{ is } L(\nu)\text{-isotypical and } [N : L(\nu)] = m. \tag{6}$$

By Lemma 1.1 one has $\text{ext}(\lambda; \nu) = \max\{m \mid \mathcal{N}(\lambda; \nu; m) \neq \emptyset\}$. Note that each module $N \in \mathcal{N}(\lambda; \nu; m)$ is a quotient of $M(\lambda) = \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} L_{\mathfrak{b}}(\lambda)$. We set

$$m(\mathfrak{g}; \mathfrak{p}; \lambda; \nu) := \max\{m \mid \exists N \in \mathcal{N}(\lambda; \nu; m) \text{ which is a quotient of } \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L_{\mathfrak{p}}(\lambda)\}. \tag{7}$$

1.3.4.

Corollary. Take $h \in \mathfrak{t}$ satisfying (3) and $\lambda \neq \nu \in \mathfrak{t}^*$ with $Re(\lambda - \nu)(h) \geq 0$. One has

$$\text{ext}(\lambda; \nu) = m(\mathfrak{g}; \mathfrak{b}; \lambda; \nu) \leq \dim M(\lambda)_\nu.$$

In particular, $\text{ext}(\lambda; \nu) \neq 0$ implies $\lambda > \nu$.

1.4. Modules over $\mathfrak{g} \times \mathfrak{h}''$

Let \mathfrak{h}'' be a finite-dimensional Lie superalgebra satisfying $[\mathfrak{h}''_{\bar{0}}, \mathfrak{h}''] = 0$. Set

$$\mathfrak{t}'' := \mathfrak{h}''_{\bar{0}}, \quad \mathfrak{g}' = \mathfrak{g} \times \mathfrak{h}'', \quad \mathfrak{h}' := \mathfrak{h} \times \mathfrak{h}'', \quad \mathfrak{t}' = \mathfrak{t} \times \mathfrak{t}'', \quad \mathfrak{p}' := \mathfrak{p} \times \mathfrak{h}''.$$

Note that the triple $(\mathfrak{g}', \mathfrak{h}', h)$ satisfy the assumption (Asst) and (3). For $\lambda \in \mathfrak{t}^*$ and $\eta \in (\mathfrak{t}'')^*$ denote by $\lambda \oplus \eta$ the corresponding element in $(\mathfrak{t}')^*$. Let $C_\lambda, C_\eta, C_{\lambda \oplus \eta}$ be the corresponding $\mathfrak{h}, \mathfrak{h}''$ and \mathfrak{h}' -modules.

1.4.1.

By [19], we can choose the grading on $C_{\lambda \oplus \eta}$ is such a way that

$$C_\lambda \boxtimes C_\eta \cong \begin{cases} C_{\lambda \oplus \eta} \oplus \Pi C_{\lambda \oplus \eta} & \text{if } C_\lambda \text{ and } C_\eta \text{ are } \Pi\text{-invariant} \\ C_{\lambda \oplus \eta} & \text{otherwise.} \end{cases}$$

Moreover if C_η is not Π -invariant, then $C_{\lambda \oplus \eta}$ is Π -invariant if and only if C_λ is Π -invariant. The similar statements hold for $M_{\mathfrak{p}}(\lambda \oplus \eta)$ and for $L_{\mathfrak{p}}(\lambda \oplus \eta)$.

1.4.2.

If $\mathfrak{h}' = \mathfrak{t}'$, then C_λ, C_ν, C_η are one-dimensional and

$$\dim \text{Ext}_{\mathcal{O}}^1(L_{\mathfrak{g}}(\lambda), L_{\mathfrak{g}}(\nu)) = \dim \text{Ext}_{\mathcal{O}}^1(L_{\mathfrak{g}'}(\lambda \oplus \eta), L_{\mathfrak{g}'}(\nu \oplus \eta)).$$

By 1.4.1 the same formula holds if $\lambda > \nu$ and C_η is not Π -invariant (or if $\lambda > \nu$ and both C_ν, C_λ are not Π -invariant). If $\mathfrak{g} = \mathfrak{q}_n$, then $\text{ext}(\lambda; \nu) \neq 0$ implies that both C_λ, C_ν are either Π -invariant or not Π -invariant and the following corollary gives $\text{ext}(\lambda; \nu) = \text{ext}(\lambda \oplus \eta; \nu \oplus \eta)$.

1.4.3.

Corollary.

- (i) $\text{ext}(\lambda \oplus \eta; \nu \oplus \eta') \neq 0$ implies $\eta' = \eta$ and $\text{ext}(\lambda; \nu) \neq 0$.
- (ii) Take $\lambda > \nu$. If at least one of the following conditions holds

- C_η is not Π -invariant
- C_ν and C_λ are not Π -invariant
- C_ν and C_λ are Π -invariant

then the map $N \mapsto N \boxtimes C_\eta$ induces a bijection between the sets $\mathcal{N}(\lambda; \nu; m)$ and $\mathcal{N}(\lambda \oplus \eta; \nu \oplus \eta; m)$. One has $m(\mathfrak{g}; \mathfrak{p}; \lambda; \nu) = m(\mathfrak{g}'; \mathfrak{p}'; \lambda \oplus \eta; \nu \oplus \eta)$ and

$$\text{ext}(\lambda; \nu) = \text{ext}(\lambda \oplus \eta; \nu \oplus \eta).$$

Proof. Observe that if $\eta' \neq \eta$, then the weight $\lambda \oplus \eta - \nu \oplus \eta'$ does not lie in $\mathbb{Z}\Delta(\mathfrak{g}')$. Combining this observation with Corollary 1.3.4 we obtain $\text{ext}(\lambda \oplus \eta; \nu \oplus \eta') = 0$ if $\eta \neq \eta'$. Other assertions follow from 1.4.1. \square

1.5.

The following lemma is a slight reformulation of Lemma 6.3 in [28].

Lemma. Take $\lambda, \nu \in \mathfrak{t}^*$ with $\lambda > \nu$.

- (i) $m(\mathfrak{g}; \mathfrak{b}; \lambda; \nu) \leq m(\mathfrak{p}; \mathfrak{b}; \lambda; \nu)$ if $\nu - \lambda \in \mathbb{N}\Delta^-(\mathfrak{p})$;
- (ii) $m(\mathfrak{g}; \mathfrak{b}; \lambda; \nu) = m(\mathfrak{g}; \mathfrak{p}; \lambda; \nu)$ if $\nu - \lambda \notin \mathbb{N}\Delta^-(\mathfrak{p})$.

Proof. For a semisimple \mathfrak{t} -module N we denote by $\Omega(N)$ the set of weights of N . For each \mathfrak{g} -module M we set

$$P_\lambda(M) := \{v \in M \mid zv = \lambda(z)v\}.$$

This defines an exact functor from $\mathfrak{g} - \text{Mod}$ to $\mathfrak{g}^z - \text{Mod}$. Recall that $\mathfrak{p} = \mathfrak{g}^z \ltimes \mathfrak{m}$ (see 1.2.1). Viewing $P_\lambda(M)$ as a \mathfrak{p} -module with the zero action of \mathfrak{m} we obtain an exact functor $P_\lambda : \mathfrak{g} - \text{Mod} \rightarrow \mathfrak{p} - \text{Mod}$. By the PBW Theorem

$$P_\lambda(M(\mu)) = \begin{cases} M_{\mathfrak{p}}(\mu) & \text{if } \mu(z) = \lambda(z) \\ 0 & \text{if } (\lambda - \mu)(z) > 0 \end{cases}$$

Let us show that

$$P_\lambda(L(\mu)) = \begin{cases} L_{\mathfrak{p}}(\mu) & \text{if } \mu(z) = \lambda(z) \\ 0 & \text{if } (\lambda - \mu)(z) > 0. \end{cases} \tag{8}$$

Indeed, since P_λ is exact, $P_\lambda(L(\mu))$ is a quotient of $P_\lambda(M(\mu))$; this gives the second formula. For the first formula assume that $\mu(z) = \lambda(z)$ and that E is a proper submodule of $P_\lambda(L(\mu))$. Since $P_\lambda(L(\mu))$ is a quotient of $P_\lambda(M(\mu))$ and

$$(P_\lambda(M(\mu)))_\mu = (M_{\mathfrak{p}}(\mu))_\mu = C_\mu$$

is a simple \mathfrak{h} -module, one has $\gamma < \mu$ for each $\gamma \in \Omega(E)$. Since E is a \mathfrak{p} -module and $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{p}$, we have $\mathcal{U}(\mathfrak{g})E = \mathcal{U}(\mathfrak{n}^-)E$. Therefore $(\mathcal{U}(\mathfrak{n}^-)E)_\mu = 0$, so $\mathcal{U}(\mathfrak{g})E$ is a proper \mathfrak{g} -submodule of $L(\mu)$. Hence $E = 0$, so $P_\lambda(L(\mu))$ is simple. This establishes (8).

Fix $N \in \mathcal{N}(\lambda; \nu; m)$ where $m := m(\mathfrak{g}; \mathfrak{b}; \lambda; \nu)$.

For (i) consider the case when $\nu - \lambda \in \mathbb{N}\Delta^-(\mathfrak{p})$. Then $\lambda(z) = \nu(z)$, so (8) gives

$$P_\lambda(L(\nu)) = L_{\mathfrak{p}}(\nu), \quad P_\lambda(L(\nu)) = L_{\mathfrak{p}}(\lambda).$$

Since P_λ is exact, $P_\lambda(N)$ is a quotient of $M_{\mathfrak{p}}(\lambda)$. Using (8) we conclude that the \mathfrak{p} -module $P_\lambda(N)$ lies in the set $\mathcal{N}(\lambda; \nu; m)$ (defined for \mathfrak{p} instead of \mathfrak{g}). Therefore $m \leq m(\mathfrak{p}; \mathfrak{b}; \lambda; \nu)$. This establishes (i).

For (ii) assume that $\nu - \lambda \notin \mathbb{N}\Delta^-(\mathfrak{p})$. Let us show that N is a quotient of $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L_{\mathfrak{p}}(\lambda)$. Write

$$\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L_{\mathfrak{p}}(\lambda) = M(\lambda)/J, \quad L_{\mathfrak{p}}(\lambda) = M_{\mathfrak{p}}(\lambda)/J'$$

where J (resp., J') is the corresponding submodule of $M(\lambda)$ (resp., of $M_{\mathfrak{p}}(\lambda)$). Since $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ is exact and $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_{\mathfrak{p}}(\lambda) = M(\lambda)$ one has $J \cong \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} J'$; in particular, each maximal element in $\Omega(J)$ lies in $\Omega(J')$. Note that

$$\Omega(J') \subset \lambda + \mathbb{N}\Delta^-(\mathfrak{p}).$$

Let $\phi : M(\lambda) \rightarrow N$ be the canonical surjection. Since $J_\lambda = 0$, $\phi(J)$ is a proper submodule of N , so $\phi(J)$ is a submodule of $\text{Soc}(N) = L(\nu)^{\oplus m}$.

If $\phi(J) \neq 0$, then ν is a maximal element in $\Omega(J)$ and so $\nu \in \lambda + \mathbb{N}\Delta^-(\mathfrak{p})$ which contradicts to $\nu - \lambda \notin \mathbb{N}\Delta^-(\mathfrak{p})$. Therefore $\phi(J) = 0$, so ϕ induces a map

$$\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L_{\mathfrak{p}}(\lambda) = M(\lambda)/J \rightarrow N.$$

Hence N is a quotient of $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L_{\mathfrak{p}}(\lambda)$ which gives $m \leq m(\mathfrak{g}; \mathfrak{p}; \lambda; \nu)$. Since $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L_{\mathfrak{p}}(\lambda)$ is a quotient of $M(\lambda)$, we have $m(\mathfrak{g}; \mathfrak{b}; \lambda; \nu) \geq m(\mathfrak{g}; \mathfrak{p}; \lambda; \nu)$. Thus $m(\mathfrak{g}; \mathfrak{b}; \lambda; \nu) = m(\mathfrak{g}; \mathfrak{p}; \lambda; \nu)$ as required. \square

1.6.

Take $z_1, \dots, z_{k-1} \in \mathfrak{t}$ satisfying (4) and the condition $\mathfrak{g}^{z_i} \subset \mathfrak{g}^{z_{i+1}}$. Setting $\mathfrak{g}^{(i)} := \mathfrak{g}^{z_i}$ we obtain the chain

$$\mathfrak{h} =: \mathfrak{g}^{(0)} \subset \mathfrak{g}^{(1)} \subset \mathfrak{g}^{(2)} \subset \dots \subset \mathfrak{g}^{(k)} := \mathfrak{g}. \tag{9}$$

We introduce $\tilde{\mathfrak{p}}^{(i)} := \mathfrak{g}^{(i)} + \mathfrak{b}$ and $\mathfrak{p}^{(i)} := \tilde{\mathfrak{p}}^{(i-1)} \cap \mathfrak{g}^{(i)}$ with $\tilde{\mathfrak{p}}^{(0)} = \mathfrak{p}^{(0)} := \mathfrak{b}$; note that $\mathfrak{p}^{(i)}$ (resp., $\tilde{\mathfrak{p}}^{(i)}$) is a parabolic subalgebra in $\mathfrak{g}^{(i)}$ (resp., in \mathfrak{g}).

1.6.1.

Taking $z_0 := h$ as in (3) and $z_k := 0$ we obtain for $s = 1, \dots, k$

$$\begin{aligned} \tilde{\mathfrak{p}}^{(s-1)} &= \sum_{\alpha: \alpha(z_{s-1}) \geq 0} \mathfrak{g}_\alpha \subset \tilde{\mathfrak{p}}^{(s)} = \sum_{\alpha: \alpha(z_s) \geq 0} \mathfrak{g}_\alpha, \\ \mathfrak{p}^{(s)} &= \sum_{\substack{\alpha: \alpha(z_{s-1}) \geq 0 \\ \alpha(z_s) = 0}} \mathfrak{g}_\alpha. \end{aligned}$$

In particular, $\tilde{\mathfrak{p}}^{(s)} = \mathfrak{g}^{(s)} \ltimes \tilde{\mathfrak{m}}^{(s)}$ and $\mathfrak{p}^{(s)} = \mathfrak{g}^{(s-1)} \ltimes \mathfrak{m}^{(s)}$ where

$$\tilde{\mathfrak{m}}^{(s)} := \sum_{\alpha: \alpha(z_s) > 0} \mathfrak{g}_\alpha, \quad \mathfrak{m}^{(s)} := \sum_{\substack{\alpha: \alpha(z_{s-1}) > 0 \\ \alpha(z_s) = 0}} \mathfrak{g}_\alpha.$$

One has $\tilde{\mathfrak{m}}^{(i+1)} \subset \tilde{\mathfrak{m}}^{(i)}$ (since $\mathfrak{g}^{(i)} \cap \mathfrak{n}$ can be identified with $\mathfrak{n}/\tilde{\mathfrak{m}}^{(i)}$).

1.6.2.

Corollary. *For $\lambda > \nu$ one has*

$$\text{ext}(\lambda; \nu) \leq m(\mathfrak{g}^{(s)}; \mathfrak{p}^{(s)}; \lambda; \nu) = \text{ext}_{\mathfrak{g}^{(s)}}(\lambda; \nu)$$

where s is minimal such that $\nu - \lambda \in \mathbb{N}\Delta^-(\mathfrak{g}^{(s)})$.

Proof. Combining Corollary 1.3.4 and Lemma 1.5 we obtain

$$\text{ext}(\lambda; \nu) = m(\mathfrak{g}; \mathfrak{b}; \lambda; \nu) \leq m(\tilde{\mathfrak{p}}^{(s)}; \tilde{\mathfrak{p}}^{(s-1)}; \lambda; \nu).$$

The $\tilde{\mathfrak{p}}^{(s)}$ -module $N := \text{Ind}_{\tilde{\mathfrak{p}}^{(s-1)}}^{\tilde{\mathfrak{p}}^{(s)}} L_{\tilde{\mathfrak{p}}^{(s-1)}}(\lambda)$ is generated by its highest weight space N_λ . Since $\tilde{\mathfrak{m}}^{(s)} \subset \mathfrak{n}$ is an ideal of $\mathfrak{p}^{(s)}$, $N^{\tilde{\mathfrak{m}}^{(s)}}$ is a submodule of N , which implies $\tilde{\mathfrak{m}}^{(s)}N = 0$. Hence N is a module over $\tilde{\mathfrak{p}}^{(s)}/\tilde{\mathfrak{m}}^{(s)} = \mathfrak{g}^{(s)}$. This gives

$$m(\tilde{\mathfrak{p}}^{(s)}; \tilde{\mathfrak{p}}^{(s-1)}; \lambda; \nu) = m(\mathfrak{g}^{(s)}; \tilde{\mathfrak{p}}^{(s-1)}/\tilde{\mathfrak{m}}^{(s)}; \lambda; \nu).$$

Using the formulae from 1.6.1 we see that $\mathfrak{p}^{(s)}$ is the image of $\tilde{\mathfrak{p}}^{(s-1)}$ in $\tilde{\mathfrak{p}}^{(s)}/\tilde{\mathfrak{m}}^{(s)} = \mathfrak{g}^{(s)}$. Therefore $m(\mathfrak{g}^{(s)}; \tilde{\mathfrak{p}}^{(s-1)}/\tilde{\mathfrak{m}}^{(s)}; \lambda; \nu) = m(\mathfrak{g}^{(s)}; \mathfrak{p}^{(s)}; \lambda; \nu)$ and thus

$$\text{ext}(\lambda; \nu) \leq m(\mathfrak{g}^{(s)}; \mathfrak{p}^{(s)}; \lambda; \nu). \tag{10}$$

Clearly, $m(\mathfrak{g}^{(s)}; \mathfrak{p}^{(s)}; \lambda; \nu) \leq \text{ext}_{\mathfrak{g}^{(s)}}(\lambda; \nu)$. Using (10) for $\mathfrak{g}^{(s)}$ we obtain

$$\text{ext}_{\mathfrak{g}^{(s)}}(\lambda; \nu) \leq m(\mathfrak{g}^{(s)}; \mathfrak{p}^{(s)}; \lambda; \nu).$$

Thus $\text{ext}_{\mathfrak{g}^{(s)}}(\lambda; \nu) = m(\mathfrak{g}^{(s)}; \mathfrak{p}^{(s)}; \lambda; \nu)$. Now (10) gives the required formula. \square

1.7. Functors $\Gamma_{\bullet}^{\mathfrak{g}, \mathfrak{p}}$

For a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ and a finite-dimensional \mathfrak{p} -module V we denote by $\Gamma^{\mathfrak{g}, \mathfrak{p}}(V)$ a maximal finite-dimensional quotient of $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(V)$. It is easy to see that this quotient is unique and that for any finite-dimensional quotient N of $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(V)$ there exists an epimorphism $\Gamma^{\mathfrak{g}, \mathfrak{p}}(V) \rightarrow N$.

1.7.1.

In [29], I. Penkov introduced important functors from $\text{Fin}(\mathfrak{p})$ to $\text{Fin}(\mathfrak{g})$. We will use a modification of these functors which appeared in [22] and other papers. These functors $\Gamma_{\bullet} = \{\Gamma_i\}_0^{\infty}$ have the following properties

- $\Gamma_0^{\mathfrak{g}, \mathfrak{p}}(V) = \Gamma^{\mathfrak{g}, \mathfrak{p}}(V)$;
- Each short exact sequence of \mathfrak{p} -modules

$$0 \rightarrow U \rightarrow V \rightarrow U' \rightarrow 0$$

induces a long exact sequence

$$\dots \rightarrow \Gamma_1^{\mathfrak{g}, \mathfrak{p}}(V) \rightarrow \Gamma_1^{\mathfrak{g}, \mathfrak{p}}(U') \rightarrow \Gamma_0^{\mathfrak{g}, \mathfrak{p}}(U) \rightarrow \Gamma_0^{\mathfrak{g}, \mathfrak{p}}(V) \rightarrow \Gamma_0^{\mathfrak{g}, \mathfrak{p}}(U') \rightarrow 0.$$

Until the end of this section we assume the existence of Γ_{\bullet} satisfying the above properties. Observe that $[\Gamma_0^{\mathfrak{g}, \mathfrak{p}}(L_{\mathfrak{p}}(\lambda)) : L_{\mathfrak{g}}(\lambda)] = 1$ if $\lambda \notin P^+(\mathfrak{g})$; we set

$$K^j(\lambda; \nu) := [\Gamma_j^{\mathfrak{g}, \mathfrak{p}}(L_{\mathfrak{p}}(\lambda)) : L_{\mathfrak{g}}(\nu)] - \delta_{0j} \delta_{\lambda\nu}.$$

Observe that $\Gamma^{\mathfrak{g}, \mathfrak{p}}(L_{\mathfrak{p}}(\lambda)) = 0$ if $\lambda \notin P^+(\mathfrak{g})$; for $\lambda \in P^+(\mathfrak{g})$ one has

$$\text{coSoc } \Gamma^{\mathfrak{g}, \mathfrak{p}}(L_{\mathfrak{p}}(\lambda)) = L_{\mathfrak{g}}(\lambda) \quad K^0(\lambda; \mu) = [\text{Rad}(\Gamma^{\mathfrak{g}, \mathfrak{p}}(L_{\mathfrak{p}}(\lambda))) : L_{\mathfrak{g}}(\mu)]. \tag{11}$$

In particular, $K^0(\lambda; \nu) \neq 0$ implies $\nu < \lambda$.

1.7.2.

Lemma. Let $\lambda, \nu \in P^+(\mathfrak{g})$ with $\nu < \lambda$ be such that

$$\forall \mu \neq \nu \quad K^0(\lambda; \mu) \neq 0 \implies \text{ext}(\mu; \nu) = 0.$$

If $K^0(\lambda; \nu) = 1$ or $\text{ext}(\nu; \nu) = 0$, then $K^0(\lambda; \nu) \leq \text{ext}(\lambda; \nu)$.

Proof. By (11) in both cases the isotypical component of $L_{\mathfrak{g}}(\nu)$ is a direct summand of $\text{Rad } \Gamma^{\mathfrak{g}, \mathfrak{p}}(L_{\mathfrak{p}}(\lambda))$, so Lemma 1.1 (i) gives the required inequality. \square

1.7.3.

Retain notation of 1.2.1 and recall that $\mathfrak{p} = \mathfrak{g}^z \ltimes \mathfrak{m}$.

Lemma. *Let $\lambda, \nu \in P^+(\mathfrak{g})$ with $\nu < \lambda$ be such that*

$$\begin{aligned} (a) \quad & K^1(\lambda; \nu) = 0 \\ (b) \quad & \forall \mu \quad K^0(\lambda; \mu) \neq 0 \implies \text{ext}(\nu; \mu) = 0. \end{aligned}$$

Then $\text{ext}_{\mathfrak{g}^z}(\lambda; \nu) \leq \text{ext}(\lambda; \nu)$.

Proof. Without loss of generality we will assume that $m := \text{ext}_{\mathfrak{g}^z}(\lambda; \nu) > 0$. By Lemma 1.1 there exists an indecomposable \mathfrak{g}^z -module N_1 with a short exact sequence

$$0 \rightarrow L_{\mathfrak{g}^z}(\nu)^{\oplus m} \rightarrow N_1 \rightarrow L_{\mathfrak{g}^z}(\lambda) \rightarrow 0$$

where $L_{\mathfrak{g}^z}(\nu)^{\oplus m}$ stands for the direct sum of m_0 copies of $L_{\mathfrak{g}^z}(\nu)$ and m_1 copies of $\Pi L_{\mathfrak{g}^z}(\nu)$ with $m_0 + m_1 = m$. Since $\mathfrak{p} = \mathfrak{g}^z \ltimes \mathfrak{m}$, the corresponding \mathfrak{p} -module $N_2 := \text{Res}_{\mathfrak{p}}^{\mathfrak{g}^z} N_1$ is an indecomposable module with a short exact sequence

$$0 \rightarrow L_{\mathfrak{p}}(\nu)^{\oplus m} \rightarrow N_2 \rightarrow L_{\mathfrak{p}}(\lambda) \rightarrow 0.$$

Consider the corresponding long exact sequence of \mathfrak{g} -modules

$$\dots \rightarrow \Gamma_1^{\mathfrak{g}, \mathfrak{p}}(L_{\mathfrak{p}}(\lambda))^{\oplus m} \xrightarrow{\phi} \Gamma_0^{\mathfrak{g}, \mathfrak{p}}(L_{\mathfrak{p}}(\nu))^{\oplus m} \rightarrow \Gamma_0^{\mathfrak{g}, \mathfrak{p}}(N_2) \rightarrow \Gamma_0^{\mathfrak{g}, \mathfrak{p}}(L_{\mathfrak{p}}(\lambda)) \rightarrow 0.$$

Recall that $\text{coSoc} \Gamma_0^{\mathfrak{g}, \mathfrak{p}}(L_{\mathfrak{p}}(\nu)) = L_{\mathfrak{g}}(\nu)$. Since $K^1(\lambda; \nu) = 0$ the image of ϕ lies in $\text{Rad} \Gamma_0^{\mathfrak{g}, \mathfrak{p}}(L_{\mathfrak{p}}(\nu))^{\oplus m}$. Thus $\Gamma_0^{\mathfrak{g}, \mathfrak{p}}(N_2)$ has a quotient N_3 with the short exact sequence

$$0 \rightarrow L_{\mathfrak{g}}(\nu)^{\oplus m} \rightarrow N_3 \rightarrow \Gamma_0^{\mathfrak{g}, \mathfrak{p}}(L_{\mathfrak{p}}(\lambda)) \rightarrow 0. \tag{12}$$

Since N_2 is indecomposable, it is generated by its λ -weight space $(N_2)_{\lambda}$. Since N_3 is a quotient of $\Gamma_0^{\mathfrak{g}, \mathfrak{p}}(N_2)$ which is a quotient of $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(N_2)$, N_3 is also generated by its λ -weight space. Hence N_3 is indecomposable and

$$\text{coSoc}(N_3) \cong L_{\mathfrak{g}}(\lambda) \cong \text{coSoc}(\Gamma_0^{\mathfrak{g}, \mathfrak{p}}(L_{\mathfrak{p}}(\lambda))).$$

The short exact sequence (12) induces a short exact sequence

$$0 \rightarrow L_{\mathfrak{g}}(\nu)^{\oplus m} \rightarrow \text{Rad}(N_3) \rightarrow M \rightarrow 0,$$

where $M := \text{Rad}(\Gamma_0^{\mathfrak{g}, \mathfrak{p}}(L_{\mathfrak{p}}(\lambda)))$. This sequence splits since, the assumption (b) gives $\text{ext}(\nu; \mu) = 0$ if $L_{\mathfrak{g}}(\mu)$ is a subquotient of M . Hence M is a submodule of N_3 , which gives the following short exact sequence

$$0 \rightarrow L_{\mathfrak{g}}(\nu)^{\oplus m} \rightarrow N_3/M \rightarrow L_{\mathfrak{g}}(\lambda) \rightarrow 0.$$

By above, N_3/M is generated by its λ -weight space, so N_3/M is indecomposable. Lemma 1.1 (i) gives $\text{ext}(\lambda; \nu) \geq m$ as required. \square

1.8.

Retain notation and assumption of 1.6. In all formulae where (s) appears, s is assumed to be one of the numbers $1, \dots, k$. For each s we fix a decomposition $\mathfrak{g}^{z_s} = \mathfrak{g}_{(s)} \times \mathfrak{h}_{(s)}^{\perp}$ in such a way that $\mathfrak{h}_{(s)}^{\perp} \subset \mathfrak{h}$ and

$$\mathfrak{g}_{(0)} \subset \mathfrak{g}_{(1)} \subset \dots \subset \mathfrak{g}_{(k)}.$$

We set $\mathfrak{h}_{(s)} := \mathfrak{g}_{(s)} \cap \mathfrak{h}$, $\mathfrak{t}_{(s)} := \mathfrak{g}_{(s)} \cap \mathfrak{t}$, $\mathfrak{t}_{(s)}^{\perp} := \mathfrak{h}_{(s)}^{\perp} \cap \mathfrak{t}$ and $\mathfrak{p}_{(s)} := (\mathfrak{g}_{(s-1)} + \mathfrak{b}) \cap \mathfrak{g}_{(s)}$. Note that $\mathfrak{p}_{(s)}$ is a parabolic subalgebra in $\mathfrak{g}_{(s)}$; one has

$$\mathfrak{g}^{z_s} = \mathfrak{g}_{(s)} + \mathfrak{h}, \quad \mathfrak{h} = \mathfrak{h}_{(s)} \times \mathfrak{h}_{(s)}^{\perp}, \quad \mathfrak{p}^{(s)} = \mathfrak{p}_{(s)} \times \mathfrak{h}_{(s)}^{\perp}, \quad \mathfrak{p}_{(s)} = \mathfrak{p}^{(s)} \cap \mathfrak{g}_{(s)}.$$

In the notation of 1.4 we have $P^+(\mathfrak{g}^{z_s}) = P^+(\mathfrak{g}_{(s)}) \oplus (\mathfrak{t}_{(s)}^{\perp})^*$. Observe that

$$\mathfrak{t}^* = P^+(\mathfrak{g}^{z_0}) \supset P^+(\mathfrak{g}^{z_1}) \supset P^+(\mathfrak{g}^{z_2}) \supset \dots \supset P^+(\mathfrak{g}^{z_k}) = P^+(\mathfrak{g}).$$

1.8.1.

We assume that for each s one has

- (A) for any $\lambda', \nu' \in P^+(\mathfrak{g}_{(s)})$ with $\text{ext}_{\mathfrak{g}_{(s)}}(\lambda', \nu') \neq 0$ the simple $\mathfrak{h}_{(s)}$ -modules $C_{\lambda'}, C_{\nu'}$ are either Π -invariant or not Π -invariant simultaneously;
- (B) there exists $\Gamma_{\bullet}^{\mathfrak{g}_{(s)}, \mathfrak{p}_{(s)}} : \text{Fin}(\mathfrak{p}_{(s)}) \rightarrow \text{Fin}(\mathfrak{g}_{(s)})$ satisfying the conditions 1.7.1.

Observe that (A) holds if $\mathfrak{h}_{\overline{1}} = 0$ (that is $\mathfrak{h} = \mathfrak{t}$); in addition (A) holds if $\mathfrak{g}_{(s)} \cong \mathfrak{q}_m$ (this follows from the description of the center of $\mathcal{U}(\mathfrak{q}_m)$) obtained in [34].

1.8.2.

Take $\lambda, \nu \in \mathfrak{t}^*$ and set $\lambda' := \lambda|_{\mathfrak{t}_{(s)}}$, $\nu' := \nu|_{\mathfrak{t}_{(s)}}$. We introduce

$$K_{(s)}^j(\lambda; \nu) := \begin{cases} 0 & \text{if } \lambda \notin P^+(\mathfrak{g}_{(s)}) \\ 0 & \text{if } \lambda|_{\mathfrak{t}_{(s)}^{\perp}} \neq \nu|_{\mathfrak{t}_{(s)}^{\perp}} \\ [\Gamma_j^{\mathfrak{g}_{(s)}, \mathfrak{p}_{(s)}}(L_{\mathfrak{p}_{(s)}}(\lambda')) : L_{\mathfrak{g}_{(s)}}(\nu')] & \text{if } \lambda|_{\mathfrak{t}_{(s)}^{\perp}} = \nu|_{\mathfrak{t}_{(s)}^{\perp}}. \end{cases} \tag{13}$$

Note that

$$K_{(s)}^0(\lambda; \nu) \neq 0 \implies \nu \in \lambda + \mathbb{N}\Delta^-(\mathfrak{g}_{(s)}), \quad \nu|_{\mathfrak{t}_{(s)}} \in P^+(\mathfrak{g}_{(s)}). \tag{14}$$

We set

$$\text{ext}_{(s)}(\lambda; \nu) := \text{ext}_{\mathfrak{g}^{z_s}}(\lambda; \nu).$$

Combining the assumption (A) and Corollary 1.4.3 we get for $\nu < \lambda$

$$\text{ext}_{(s)}(\lambda; \nu) = \begin{cases} 0 & \text{if } \lambda|_{\mathfrak{t}_{(s)}^\perp} \neq \nu|_{\mathfrak{t}_{(s)}^\perp} \\ \text{ext}_{\mathfrak{g}_{(s)}}(\lambda'; \nu') & \text{if } \lambda|_{\mathfrak{t}_{(s)}^\perp} = \nu|_{\mathfrak{t}_{(s)}^\perp} \end{cases} \tag{15}$$

Note that $m(\mathfrak{g}^{(s)}; \mathfrak{p}^{(s)}; \lambda; \nu) \leq K_{(s)}^0(\lambda; \nu)$.

1.9. Graph $G(\mathfrak{t}^*; K^0)$

For $\lambda, \nu \in \mathfrak{t}^*$ we introduce

$$s(\lambda; \nu) := \max\{s \mid \lambda|_{\mathfrak{t}_{(s)}^\perp} = \nu|_{\mathfrak{t}_{(s)}^\perp}\}, \quad k_0(\lambda; \nu) := K_{(s(\lambda; \nu))}^0(\lambda; \nu),$$

Note that $s(\lambda; \nu) = \min\{s \mid \nu - \lambda \in \mathbb{N}\Delta^-(\mathfrak{g}_{z_s})\}$ if $\nu < \lambda$. Corollary 1.6.2 gives

$$\text{ext}(\lambda; \nu) \leq \text{ext}_{(s(\lambda; \nu))}(\lambda; \nu) \leq k_0(\lambda; \nu) \text{ for each } \lambda, \nu \in P^+(\mathfrak{g}) \text{ with } \nu < \lambda. \tag{16}$$

1.9.1. Definitions

We say that $(\lambda; \nu)$ is K^i -stable if $K_{(s)}^i(\lambda; \nu) \neq 0$ for each $s > s(\lambda; \nu)$.

Let $G(\mathfrak{t}^*; K^0)$ be the graph with the set of vertices \mathfrak{t}^* connected by $k_0(\lambda; \nu)$ -edges of the form $\nu \rightarrow \lambda$.

For each $B \subset \mathfrak{t}^*$ we denote by $G(B, K^0)$ the induced subgraph of $G(\mathfrak{t}^*; K^0)$. We say that a graph $G(B, K^0)$ is *bipartite* if there exists $\text{dex} : B \rightarrow \mathbb{Z}_2$ such that $\nu \rightarrow \lambda$ implies $\text{dex}(\nu) \neq \text{dex}(\lambda)$. For each $\lambda \in \mathfrak{t}^*$ let $B(\lambda)$ be the set consisting of λ and all its direct predecessor in $G(\mathfrak{t}^*; K^0)$, i.e.

$$B(\lambda) := \{\lambda\} \cup \{\nu \mid k_0(\lambda; \nu) \neq 0\}.$$

1.9.2. Remarks

Observe that $G(\mathfrak{t}^*; K^0)$ is a directed graph without cycles (for any edge $\mu \rightarrow \nu$ one has $\mu < \nu$). For $\mathfrak{g} \neq \mathfrak{gl}(n|n)$ one has $B(0) = \{0\}$ since 0 is a minimal weight in $P^+(\mathfrak{g}_{(i)})$ for each i .

Note that if $G(B(\lambda); K^0)$ is bipartite and $\text{ext}(\nu; \nu) = 0$ for each $\nu \in B(\lambda)$, then the radical of $\Gamma^{\mathfrak{g} \cdot \mathfrak{p}} L_{\mathfrak{p}}(\lambda)$ is semisimple.

1.9.3.

Corollary. *Let $\lambda \in P^+(\mathfrak{g})$ be such that*

- (a) $B(\lambda) \subset P^+(\mathfrak{g})$ and $G(B(\lambda); K^0)$ is bipartite;
- (b) $\text{ext}_{(s)}(\mu; \nu) = 0 \iff \text{ext}_{(s)}(\nu; \mu) = 0$ for all s and $\mu, \nu \in B(\lambda) \setminus \{\lambda\}$.

- (i) If $\nu \in P^+(\mathfrak{g})$ with $\nu < \lambda$ satisfies
 - (c) $(\lambda; \nu)$ is K^1 -stable;
 - (d) $(\lambda; \nu)$ is K^0 -stable or $\text{ext}_{(s)}(\nu; \nu) = 0$ for each s ,
then $\text{ext}(\lambda; \nu) = \text{ext}_{(s_0(\lambda; \nu))}(\lambda; \nu)$.
- (ii) If λ, ν satisfy (a)–(d) and
 - (e) $k_0(\lambda; \nu) = 1$ or $\text{ext}_{(s(\lambda; \nu))}(\nu; \nu) = 0$,
then $\text{ext}(\lambda; \nu) = k_0(\lambda; \nu)$.

Proof. If $\nu \notin B(\lambda)$, then (16) gives $\text{ext}(\lambda; \nu) = k_0(\lambda; \nu) = 0$. Assume that $\nu \in B(\lambda)$. Set

$$p := s_0(\lambda; \nu).$$

Combining (15), (16) we obtain $\text{ext}_{\mathfrak{g}^{z_s}}(\mu_2; \mu_1) \leq k_0(\mu_2; \mu_1)$ for $\mu_1, \mu_2 \in P^+(\mathfrak{g})$ if $\mu_2 > \mu_1$. Then the assumptions (a), (b) give

$$\text{ext}_{(s)}(\mu_1; \mu_2) = 0 \text{ for all } \mu_1 \neq \mu_2 \in B(\lambda) \setminus \{\lambda\} \text{ and each } s. \tag{17}$$

Take $s > p$ and view λ, ν as elements of $P^+(\mathfrak{g}^{z_s})$. We will use Lemma 1.7.3 for the pair $\mathfrak{p}^{(s)} \subset \mathfrak{g}^{z_s}$. Let us check the assumptions of this lemma: the assumption (a) follows from (c) and the assumption (b) follows from (17) for $\mu \neq \nu$ (since $\nu \in B(\lambda)$); the assumption (b) for $\mu = \nu$ means that $K_{(s)}^0(\lambda; \nu) = 0$ implies $\text{ext}_{(s)}(\nu; \nu) = 0$ — this follows from (d). Lemma 1.7.3 gives $\text{ext}_{(s)}(\lambda; \nu) \leq \text{ext}_{(s+1)}(\lambda; \nu)$. Using (16) we get

$$\text{ext}(\lambda; \nu) \leq \text{ext}_{(p)}(\lambda; \nu) \leq \text{ext}_{(n)}(\lambda; \nu) = \text{ext}(\lambda; \nu).$$

This proves (i). For (ii) note that (17) and (e) imply the assumptions of Lemma 1.7.2 for \mathfrak{g}^{z_p} which gives $K_p^0(\lambda; \nu) \leq \text{ext}_{(p)}(\lambda; \nu)$. By (i) this can be rewritten as $k_0(\lambda; \nu) \leq \text{ext}(\lambda; \nu)$. Now (16) gives $k_0(\lambda; \nu) = \text{ext}(\lambda; \nu)$ as required. \square

1.9.4. Remark

If λ satisfies (a), (b) and the assumption (e) holds for each $\nu \in B(\lambda)$, then $\Gamma^{\mathfrak{g}, \mathfrak{p}} L_{\mathfrak{p}}(\lambda)$ has a semisimple radical.

1.9.5. Remark

In the examples considered below each pair $(\lambda; \nu)$ with $\lambda \neq \nu$ is K^i -stable for any i (in fact $K_{(s)}^i(\lambda; \nu) \neq 0$ implies $K_{(s')}^{i'}(\lambda; \nu) \neq 0$ for each $s' \neq s$ and any i'). In most of the cases $G(B(\lambda); K^0)$ is bipartite (this simply means that $k_0(\mu_1; \mu_2) = 0$ for $\mu_1, \mu_2 \in B(\lambda) \setminus \{\lambda\}$); moreover, $K_{(s)}^i(\lambda; \nu) \neq 0$ implies $\text{dex}(\nu) \equiv \text{dex}(\lambda) + i$ modulo 2.

2. Weight diagrams and arch diagrams

In this section we introduce the language of “arch diagrams” which will be used in Section 3. We will consider the following examples

- the principal block over $\mathfrak{g} = \mathfrak{gl}(n|n), \mathfrak{osp}(2n + t|2n)$ for $t = 0, 1, 2$;
- the principal block over $\mathfrak{q}_{2n+\ell}$ for $\ell = 0, 1$;
- the “half-integral” block of maximal atypicality over \mathfrak{q}_{2n} .

We set $\ell := \dim \mathfrak{t} - 2n$, i.e., $\ell = 1$ for $\mathfrak{osp}(2n + 2|2n), \mathfrak{q}_{2n+1}$ and $\ell = 0$ in other cases.

2.1. Triangular decompositions

We fix the following bases of simple roots

$$\Sigma := \begin{cases} \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_n - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n & \text{for } \mathfrak{gl}(n|n) \\ \varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \dots, \varepsilon_n - \delta_n, \delta_n & \text{for } \mathfrak{osp}(2n + 1|2n) \\ \delta_1 - \varepsilon_1, \varepsilon_1 - \delta_2, \dots, \varepsilon_{n-1} - \delta_n, \delta_n \pm \varepsilon_n & \text{for } \mathfrak{osp}(2n|2n) \\ \varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \dots, \varepsilon_n - \delta_n, \delta_n \pm \varepsilon_{n+1} & \text{for } \mathfrak{osp}(2n + 2|2n). \\ \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{2n+\ell-1} - \varepsilon_{2n+\ell} & \text{for } \mathfrak{q}_{2n+\ell} \end{cases}$$

and take the following Weyl vector

$$\rho := \begin{cases} \sum_{i=1}^n (n - i)(\varepsilon_i - \delta_{n+1-i}) & \text{for } \mathfrak{gl}(n|n) \\ 0 & \text{for } \mathfrak{osp}(2n|2n), \mathfrak{osp}(2n + 2|2n), \mathfrak{q}_{2n+\ell} \\ \frac{1}{2} \sum_{i=1}^n (\delta_i - \varepsilon_i) & \text{for } \mathfrak{osp}(2n + 1|2n). \end{cases}$$

2.2. Weight diagrams

We denote by \mathcal{B}_0 the set of the highest weights for simple modules lying in the principal block of $\mathcal{F}in(\mathfrak{g})$; for \mathfrak{q}_{2n} we denote by $\mathcal{B}_{1/2}$ the set of the highest weights for simple modules lying in the half-integral block of maximal atypicality. In what follows \mathcal{B} will denote \mathcal{B}_0 or $\mathcal{B}_{1/2}$. These sets can be described as follows.

- For $\mathfrak{gl}(n|n)$ the set \mathcal{B}_0 consists of λ s such that $\lambda + \rho = \sum_{i=1}^n \lambda_i(\varepsilon_i - \delta_i)$, where $\lambda_1, \dots, \lambda_n$ are integers with $\lambda_{i+1} < \lambda_i$.

- For $\mathfrak{osp}(2n + t|2n)$ the set \mathcal{B}_0 consists of λ s such that

$$\lambda + \rho = \begin{cases} \sum_{i=1}^{n-1} \lambda_i(\varepsilon_i + \delta_i) + \lambda_n(\delta_n + \xi\varepsilon_n) & \text{for } t = 0 \\ \sum_{i=1}^n \lambda_i(\varepsilon_i + \delta_i) & \text{for } t = 2 \\ \sum_{i=1}^{s-1} (\lambda_i + \frac{1}{2})(\varepsilon_i + \delta_i) + \frac{1}{2}(\delta_s + \xi\varepsilon_s) + \sum_{i=s+1}^n \frac{1}{2}(\delta_i - \varepsilon_i) & \text{for } t = 1 \end{cases}$$

where $\xi \in \{\pm 1\}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{N}$ with $\lambda_{i+1} < \lambda_i$ or $\lambda_i = \lambda_{i+1} = 0$. For $t = 1$ we have $1 \leq s \leq n + 1$ and we set $\lambda_s := \lambda_{s+1} := \dots = \lambda_n = 0$ if $s \leq n$ (for $s = n + 1$ we have $\lambda + \rho = \sum_{i=1}^n (\lambda_i + \frac{1}{2})(\varepsilon_i + \delta_i)$).

- For $\mathfrak{q}_{2n+\ell}$ the set \mathcal{B}_0 consists of λ s such that

$$\lambda + \rho = \sum_{i=1}^n \lambda_i(\varepsilon_i - \varepsilon_{2n+\ell+1-i}),$$

where $\lambda_1, \dots, \lambda_n \in \mathbb{N}$ with $\lambda_{i+1} < \lambda_i$ or $\lambda_{i+1} = \lambda_i = 0$.

- For \mathfrak{q}_{2n} the set $\mathcal{B}_{1/2}$ consists of λ s such that

$$\lambda + \rho = \sum_{i=1}^n \lambda_i(\varepsilon_i - \varepsilon_{2n+1-i}),$$

where $\lambda_1, \dots, \lambda_n \in \mathbb{N} + 1/2$ and $\lambda_{i+1} < \lambda_i$.

2.2.1.

We assign to λ as above a “weight diagram”: for \mathcal{B}_0 (resp., $\mathcal{B}_{1/2}$) the weight diagram is a number line with one or several symbols drawn at each position with integral (resp., half-integral) coordinate:

- we put the sign \times at each position with the coordinate λ_i ;
- if $\ell = 1$ we add $>$ at the zero position;
- we add the “empty symbol” \circ to all empty positions;
- for $\mathfrak{osp}(2n|2n)$ with $\lambda_k \neq 0$ and for $\mathfrak{osp}(2n + 1|2n)$ with $s \leq k$, we write the sign of ξ before the diagram (+ if $\xi = 1$ and $-$ if $\xi = -1$).

Note that $\lambda \in \mathcal{B}_0$ (resp., $\lambda \in \mathcal{B}_{1/2}$) is uniquely determined by the weight diagram constructed by the above procedure.

For a diagram f we denote by $f(a)$ the symbols at the position a (for $\mathfrak{gl}(n|n)$ one has $f(a) \in \{\circ, \times\}$). For $\mathfrak{osp}(2n|2n)$ (resp., $\mathfrak{osp}(2n + 1|2n)$) a diagram has a sign if and only if $f(0) = \circ$ (resp., $f(0) \neq \circ$). We say that two weight diagrams “have different signs” if one of them has sign $+$ and another sign $-$.

2.2.2.

Consider the case $\mathfrak{g} \neq \mathfrak{gl}(n|n)$. In this case each position with negative coordinate contains \circ and we will not depict these positions. Each position with a positive coordinate contains either \times or \circ . For $\ell = 0$ the zero position is occupied either by \circ or by several symbols \times ; we write this as \times^i for $i \geq 0$. Similarly, for $\ell = 1$ the zero position is occupied by $>$ with $i \geq 0$.

2.2.3. Examples

For $\mathfrak{gl}(3|3)$ the weight diagram of 0 is $\dots \circ \circ \times \times \times \circ \circ \dots$, where the leftmost \times occupies the zero position. The weight diagram of 0 is

$$\begin{aligned} & \times^n \circ \circ \dots && \text{for } \mathfrak{osp}(2n|2n), \mathfrak{q}_{2n} \\ - & \times^n \circ \circ \dots && \text{for } \mathfrak{osp}(2n+1|2n) \\ & \begin{matrix} \times^n \\ > \end{matrix} \circ \circ \dots && \text{for } \mathfrak{osp}(2n+2|2n), \mathfrak{q}_{2n+1}. \end{aligned}$$

The diagram $+\circ \times$ corresponds to the $\mathfrak{osp}(4|4)$ -weight $\lambda = \lambda + \rho = (\varepsilon_2 + \delta_2) + 2(\varepsilon_1 + \delta_1)$. The diagram $+\times^3$ corresponds to $\mathfrak{osp}(7|6)$ -weight $\lambda = \varepsilon_1$.

The empty diagram corresponds to one of the algebras $\mathfrak{gl}(0|0) = \mathfrak{osp}(0|0) = \mathfrak{osp}(1|0) = \mathfrak{q}_0 = 0$; the diagram $>$ corresponds to the weight 0 for $\mathfrak{osp}(2|0) = \mathbb{C}$ or for \mathfrak{q}_1 (in both cases the corresponding simple highest weight module is one-dimensional).

2.2.4. Remark

By [8], Proposition 4.11 for $\lambda \in \mathcal{B}_0$ the simple $OSP(2n|2n)$ -module is either of the form $L(\lambda)$ if $\lambda_n = 0$ or $L(\lambda) \oplus L(\lambda^\sigma)$, where λ^σ is obtained from λ by changing the sign of ξ . Thus the simple $OSP(2n|2n)$ -modules are in one-to-one correspondence with the unsigned $\mathfrak{osp}(2n|2n)$ -diagrams.

2.3. Arch diagrams

A *generalized arch diagram* is the following data:

- a weight diagram f , where the symbols \times at the zero position are drawn vertically and $>$ (if it is present) is drawn in the bottom,
- a collection of non-intersecting arches, where each arch is
 - either $\text{arc}(a; b)$ connecting the symbol \times with \circ at the position $b > a$;
 - or $\text{arc}(0; b, b')$ connecting the symbol \times at the zero position with two symbols \circ at the positions $0 < b < b'$;
 - for \mathfrak{q}_{2n+1} -case $\text{arc}(0; b)$ connecting $>$ (at the zero position) with \circ at the positions $b > 0$; this arch is called *wobbly*.⁴

⁴ Wobbly arches are important for the description of $\text{DS}_x(L)$; we will not use them in our text.

An empty position is called *free* if this position is not an end of an arch; we say that $\text{arc}(a; b)$ is a *two-legged arch originated at a* and $\text{arc}(0; b, b')$ is a *three-legged arch originated at 0*. A generalized arch diagram is called *arch diagram* if

- each symbol \times is the left end of exactly one arch;
- for \mathfrak{q}_{2n+1} -case the symbol $>$ is the left end of a wobbly arch;
- there are no free positions under the arches;
- for the \mathfrak{gl} -case all arches are two-legged;
- for the $\mathfrak{osp}(2n|2n)$, $\mathfrak{osp}(2n + 1|2n)$ -cases the lowest \times at the zero position supports a two-legged arch and each other symbol \times at the zero position supports a three-legged arch;
- for the $\mathfrak{q}_{2n+\ell}$, $\mathfrak{osp}(2n + 2|2n)$ -cases each symbol \times at the zero position supports a three-legged arch.

Each weight diagram f admits a unique arch diagram which we denote by $\text{Arc}(f)$; this diagram can be constructed in the following way: we pass from right to left through the weight diagram and connect each symbol \times with the next empty symbol(s) to the right by an arch. Examples of arch (arc) diagrams in $\mathfrak{gl}(m|n)$ and \mathfrak{osp} -cases appear in [23], Sect. 4 and in [15], 8.1.3, 8.1.4. For \mathfrak{q}_n -case slightly different arch (arc) diagrams were introduced in [19] (see 4.3.3 in [19] for pictures). The arc diagrams in [19] are obtained from our arch diagrams by the following procedure: a wobbly arch (if it appears) is substituted by two “half arcs” and each three-legged arch is substituted by two “half arcs”.

2.3.1. Partial order

We consider a partial order on the set of arches by saying that one arch is smaller than another one if the first one is “below” the second one:

$$\begin{aligned} \text{arc}(a; b) > \text{arc}(a'; b') &\iff a < a' < b \\ \text{arc}(0; b_1, b_2) > \text{arc}(a'; b') &\iff a' < b_2, \\ \text{arc}(0; b_1, b_2) > \text{arc}(0; b'_1, b'_2) &\iff b_2 > b'_2 \iff b_1 > b'_1. \end{aligned}$$

2.4. Map τ

Following [22], we introduce a bijection τ between the weight diagrams for $\mathfrak{osp}(2n + 2|2n)$ and $\mathfrak{osp}(2n + 1|2n)$: for a $\mathfrak{osp}(2n + 2|2n)$ -diagram f we construct $\tau(f)$ by the following procedure:

- we remove $>$ and then shift all entires at the non-zero positions of f by one position to the left;
- we add the sign $+$ if $f(1) = \times$ and the sign $-$ if $f(1) = \circ$ and $f(0) \neq >$.

For instance, $\tau(\overset{\times}{>}) = -\times$, $\tau(>\times) = +\times$, $\tau(\overset{\times}{>}\circ\times) = -\times\times$, $\tau(>\circ\times) = \circ\times$.

One readily sees that τ is a one-to-one correspondence between weight diagrams and that there is a natural bijection between the arches in $\text{Arc}(f)$ and $\text{Arc}(\tau(f))$: the image of $\text{arc}(a; b)$ is $\text{arc}(a - 1; b - 1)$, the image of $\text{arc}(0; b_1, b_2)$ is $(0; b_1 - 1, b_2 - 1)$ if $b_1 \neq 0$ and $(0; b_2 - 1)$ if $b_1 = 0$; this bijection preserves the partial order of the arches.

We will also denote by τ the corresponding bijection between the weight (i.e., between the sets \mathcal{B}_0 defined for $\mathfrak{osp}(2n + 2|2n)$ and $\mathfrak{osp}(2n + 1|2n)$).

2.5. The algebras $\mathfrak{g}_{(s)}$

For $\mathfrak{g} = \mathfrak{osp}(2n + t|2n)$ we consider the chain

$$\mathfrak{osp}(t|0) \subset \mathfrak{osp}(2 + t|2) \subset \mathfrak{osp}(4 + t|4) \subset \dots \subset \mathfrak{osp}(2n + t|2n) = \mathfrak{g}$$

where $\mathfrak{osp}(2p + t|2p)$ corresponds to the last $2p + \lfloor \frac{t}{2} \rfloor$ roots in Σ ; we denote the subalgebra $\mathfrak{osp}(2s + t|2s)$ by $\mathfrak{g}_{(s)}$. Note that $\mathfrak{g}_{(0)} = 0$ for $t = 0, 1$ and $\mathfrak{g}_{(0)} = \mathbb{C}$ to $t = 2$.

Similarly, for $\mathfrak{g} = \mathfrak{gl}(n|n), \mathfrak{q}_{n+\ell}$ we consider the chains

$$0 = \mathfrak{gl}(0|0) \subset \mathfrak{gl}(1|1) \subset \dots \subset \mathfrak{gl}(n|n) \quad \mathfrak{q}_\ell \subset \mathfrak{q}_{2+\ell} \subset \dots \subset \mathfrak{q}_{2n+\ell}$$

where for $i > 0$ the subalgebras $\mathfrak{gl}(i|i)$ (resp., $\mathfrak{q}_{2i+\ell}$) corresponds to the middle $2i + \ell - 1$ roots in Σ ; we denote the subalgebra $\mathfrak{gl}(s|s)$ (resp., $\mathfrak{q}_{2s+\ell}$) by $\mathfrak{g}_{(s)}$. It is easy to see that for each s there exist $z_s \in \mathfrak{t}$ such that $\mathfrak{g}^{z_s} = \mathfrak{g}_{(s)} + \mathfrak{h}$.

2.5.1.

We retain notation of 1.8. For $\lambda \in \mathfrak{t}^*$ we denote by $\text{tail}(\lambda)$ the maximal i such that $\lambda|_{\mathfrak{t}^{(i)}} = 0$. If $\rho = 0$ (i.e., for $\mathfrak{g} = \mathfrak{osp}(2n|2n), \mathfrak{osp}(2n + 2|2n)$ and $\mathfrak{q}_{2n+\ell}$) then $\text{tail}(\lambda)$ is equal to the number of \times at the zero position of the weight diagram (and is equal to the number of zeros among $\{\lambda_i\}_{i=1}^n$). The map τ defined in 2.4 preserves the function tail .

3. Multiplicities $K^i(\lambda; \nu)$

We retain notation of Section 2 and set $\mathfrak{p} := \mathfrak{g}_{(n-1)} + \mathfrak{b}$. The multiplicities $K^i(\lambda; \nu)$ were obtained in [30], [31], [28] and [22]. Below we will describe these multiplicities in terms of arch diagrams. We introduce a Poincaré polynomial $K^{\lambda, \nu}(z)$ by

$$K^{\lambda, \nu}(z) := \sum_{i=0}^{\infty} K^i(\lambda; \nu) z^i = \sum_{i=0}^{\infty} [\Gamma_i^{\mathfrak{g}, \mathfrak{p}}(L_{\mathfrak{p}}(\lambda)) : L_{\mathfrak{g}}(\nu)] z^i$$

(by [29], the sum is finite). One has $K^{\lambda, \nu}(z) = 0$ if $\lambda \in \mathcal{B}$ and $\nu \notin \mathcal{B}$. The polynomials $K^{\lambda, \nu}(z)$ for $\lambda, \nu \in \mathcal{B}$ are given in Propositions 3.2, 3.3, 3.4. Proposition 3.2 (\mathfrak{gl} -case) is a simple reformulation of Corollary 3.8 in [28]. Proposition 3.3 (\mathfrak{osp} -case) is a reformulation

of Proposition 7 in [22] (we translate the formulae from [22] to the language of arch diagrams). For the \mathfrak{q} -case the polynomials were described recursively by V. Serganova and I. Penkov in [30], [31]; in Proposition 3.4 we present non-recursive formulae, which are deduced from the Penkov-Serganova recursive formulae. The rest of the section is occupied by examples and the proof of Proposition 3.4.

3.1. Notation

Let g be a weight diagram. We denote by $(g)_p^q$ the diagram which obtained from g by moving \times from the position p to a free position $q > p$; such diagram is defined only if

$$g(p) \in \{ \times^i, \succ^{\times^i} \} \text{ for } i \geq 1 \quad \text{and} \quad g(q) = \circ.$$

For instance, for $g = \times^2 \circ \times$ one has $(g)_0^3 = \times \circ \times \times$ and $(g)_0^2, (g)_1^5$ are not defined. If $g(0) = \times^i$ or \succ^{\times^i} for $i > 1$, we denote by $(g)_{0,0}^{p,q}$ the diagram which obtained from g by moving two symbols \times from the zero position to free positions p and q with $p < q$; for example, $(\times^2 \times)_{0,0}^{3,4} = \circ \times \circ \times \times$.

If $f(p) \neq \circ$, we denote by $\text{arc}_f(p)$ the positions “connected with p in $\text{Arc}(f)$ ”; for example, $\text{arc}_{\times \circ \times}(2) = 3$, $\text{arc}_{\times \circ \times}(0) = \{1, 4\}$ for \mathfrak{q}_4 and $\text{arc}_{\times \circ \times}(0) = \{1\}$ for $\mathfrak{osp}(4|4)$. Notice that if $(f)_p^q$ is defined, then $\text{arc}_f(p)$ is defined.

We always assume that $\lambda, \nu \in \mathcal{B}$ and denote by g (resp., f) the weight diagram of λ (resp., of ν); we sometimes write $K(\frac{g}{f})$ instead of $K^{\lambda, \nu}$. As in 2.2 let λ_1 be the coordinate of the rightmost symbol \times in g .

3.2. Proposition (see [28], Corollary 3.8)

Take $\mathfrak{g} = \mathfrak{gl}(n|n)$. If $K^{\lambda, \nu}(z) \neq 0$, then $g = (f)_a^{\lambda_1}$ and

$$K^{\lambda, \nu}(z) = \begin{cases} z^{b-\lambda_1} & \text{if } \lambda_1 \leq b \\ 0 & \text{if } b < \lambda_1 \end{cases}$$

where $b := \text{arc}_f(a)$.

3.3. Proposition (see [22], Proposition 7)

Take $\mathfrak{g} = \mathfrak{osp}(2n + t|2n)$ for $t = 0, 2$ and $\lambda \neq 0$.

- (i) If $K^{\lambda, \nu}(z) \neq 0$, then $g = (f)_a^{\lambda_1}$ or $g = (f)_{0,0}^{p, \lambda_1}$ and f, g do not have different signs.
- (ii) Let $g = (f)_a^{\lambda_1}$ and f, g do not have different signs.

Set $b := \max \text{arc}_f(a)$ and $b_- := \min \text{arc}_f(0)$ if $a = 0$.

$$\text{If } a \neq 0 \text{ or } a = 0 \text{ and } t = 2, \text{ then } K^{\lambda, \nu}(z) = \begin{cases} z^{b-\lambda_1} & \text{if } \lambda_1 \leq b \\ 0 & \text{if } b < \lambda_1. \end{cases}$$

If $a = 0$ and $t = 0$, then

$$K^{\lambda, \nu}(z) = \begin{cases} z^{b_- - \lambda_1} + z^{b - \lambda_1} & \text{if } \lambda_1 \leq b_- < b \\ z^{b - \lambda_1} & \text{if } \lambda_1 \leq b_- = b \\ z^{b - \lambda_1} & \text{if } b_- < \lambda_1 \leq b \\ 0 & \text{if } b < \lambda_1. \end{cases}$$

- (iii) Let $g = (f)_{0,0}^{p,\lambda_1}$. If $\text{Arc}(f)$ contains $\text{arc}(0; p, q)$ with $\lambda_1 \leq q < \max \text{arc}_f(0)$, then $K^{\lambda, \nu}(z) = z^{q - \lambda_1}$; otherwise $K^{\lambda, \nu}(z) = 0$.
- (iv) For $\lambda \neq 0$ the polynomials $K^{\lambda, \nu}(z)$ for $\mathfrak{osp}(2n + 1|2n)$ can be obtained from the polynomials for $\mathfrak{osp}(2n + 2|2n)$ by the formula $K^{\tau(\lambda), \tau(\nu)}(z) = K^{\lambda, \nu}(z)$.

3.3.1. Examples

- (1) For $\lambda = \varepsilon_1 + \delta_1$ and $\nu = 0$ one has $g = (f)_0^1$ with $b = 2n$ for $\mathfrak{osp}(2n + 2|2n)$ and $b = 2n - 1, b_- = 1$ for $\mathfrak{osp}(2n|2n)$. The polynomial $K^{\varepsilon_1 + \delta_1, 0}(z)$ equals to 1 for $\mathfrak{osp}(2|2)$, to $1 + z^{2n-2}$ for $\mathfrak{osp}(2n|2n)$ with $n > 1$ and to z^{2n-1} for $\mathfrak{osp}(2n + 2|2n)$.
- (2) Take $\mathfrak{g} = \mathfrak{osp}(4|4)$ with $\nu = \varepsilon_1 + \delta_1$. Then $f = \times \times$ so

$$\text{Arc}(f) = \{\text{arc}(1; 2), \text{arc}(0; 3)\}, \quad \text{arc}_f(1) = \{2\}, \quad \text{arc}_f(0) = \{3\}.$$

The non-zero values of $K(\frac{g}{f})$ are given by the following table

g	$\times \circ \times = (\times \times)_1^2$	$\circ \times \times = (\times \times)_0^2$	$\circ \times \circ \times = (\times \times)_0^3$
$K(\frac{g}{f})$	1	z	1

- (3) Take $\mathfrak{g} = \mathfrak{osp}(6|4)$ with $\nu = \varepsilon_1 + \delta_1$. Then $f = \overset{\times}{>} \times$ so

$$\text{Arc}(f) = \{\text{arc}(1; 2), \text{arc}(0; 3, 4)\}, \quad \text{arc}_f(1) = \{2\}, \quad \text{arc}_f(0) = \{3, 4\}.$$

The non-zero values of $K(\frac{g}{f})$ are given by the following table

g	$\overset{\times}{>} \circ \times = (f)_1^2$	$> \times \times = (f)_0^2$	$> \times \circ \times$	$> \times \circ \circ \times = (f)_0^4$
$K(\frac{g}{f})$	1	z^2	z	1

- (4) Take $\mathfrak{g} = \mathfrak{osp}(6|6)$ with $\nu = \varepsilon_1 + \delta_1$. Then $f = \times^2 \times$ so

$$\text{Arc}(f) = \{\text{arc}(1; 2), \text{arc}(0; 3), \text{arc}(0; 4, 5)\}, \quad \text{arc}_f(1) = \{2\}, \quad \text{arc}_f(0) = \{3, 4, 5\}.$$

The non-zero values of $K(\frac{g}{f})$ are given by the following table

g	$\times^2 \circ \times$	$\times \times \times$	$\times \times \circ \times$	$\times \times \circ \circ \times$	$\times \times \circ \circ \circ \times$
$K(\frac{g}{f})$	1	$z + z^3$	$1 + z^2$	z	1

(5) Take $\mathfrak{g} = \mathfrak{osp}(10|10)$ with $f = \times^3 \circ \circ \times \times$. Then

$$\text{Arc}(f) = \{(\text{arc}(4; 5), \text{arc}(3; 6), \text{arc}(0; 1), \text{arc}(0; 2, 7), \text{arc}(0; 9, 10))\}.$$

For $g = \times^3 \circ \circ \times \circ \times$, $\times^3 \circ \circ \circ \times \circ \times$, one has $K(\frac{g}{f}) = 1$. In addition,

g	$\times \circ \times \times \times \times = (f)_{0,0}^{2,5}$	$(f)_{0,0}^{2,6}$	$\times \circ \times \times \times \circ \times \times = (f)_{0,0}^{2,7}$
$K(\frac{g}{f})$	z^2	z	1

For $g = \times^3 \circ \circ \circ \times \times$ one has $K(\frac{g}{f}) = z$; for $g = (f)_0^i$ with $i = 5, 6, \dots, 10$ we have $K(\frac{g}{f}) = z^{10-i}$. Since $K(\frac{g}{f}) \neq 0$ implies $\lambda_1 > \nu_1 = 4$ we get $K(\frac{g}{f}) = 0$ for other values of g .

3.4.

Proposition. Take $\mathfrak{g} = \mathfrak{q}_m$ and $\lambda, \nu \in \mathcal{B}_0$ or $\lambda, \nu \in \mathcal{B}_{1/2}$.

- (i) One has $K^{0,0}(z) = z + z^2 + \dots + z^{m-1}$ and $K^{\lambda,\nu}(z) = 0$ for $\nu \neq 0$.
- (ii) If $\lambda \neq 0$ and $K^{\lambda,\nu}(z) \neq 0$, then $g = (f)_a^{\lambda_1}$ for $a < \lambda_1$.
- (iii) Let $g = (f)_a^{\lambda_1}$ for $a < \lambda_1$. Set $b := \max \text{arc}_f(a)$.

$$\text{If } a \neq 0, \text{ then } K^{\lambda,\nu}(z) = \begin{cases} z^{b-\lambda_1} & \text{if } \lambda_1 \leq b \\ 0 & \text{if } b < \lambda_1. \end{cases}$$

If $a = 0$, set $A_{f;\lambda_1} := \{i \in \text{arc}_f(0) \mid \lambda_1 \leq i < b\}$. Then

$$K^{\lambda,\nu}(z) = \begin{cases} 0 & \text{if } A_{f;\lambda_1} = \emptyset \\ z^{i_--\lambda_1} + z^{i_+-\lambda_1} & \text{otherwise} \end{cases}$$

where $i_- := \min A_{f;\lambda_1}$, $i_+ := \max A_{f;\lambda_1}$.

3.4.1. Examples

In the examples below we compute $K^{\lambda,\nu}(z)$ using Proposition 3.4.

- (1) For $\lambda = \varepsilon_1 - \varepsilon_m$ and $\nu = 0$ one has $g = (f)_0^1$ with $\text{arc}_f(0) = \{1, \dots, m\}$ and thus $A_{f;1} = \{1, \dots, m-1\}$. This gives $K^{\varepsilon_1-\varepsilon_m,0} = 1 + z^{m-2}$ as in [30], Theorem 4.
- (2) Take $\mathfrak{g} = \mathfrak{q}_4$ and $f = \times \times$. Then

$$\text{Arc}(f) = \{\text{arc}(1; 2); \text{arc}(0; 3, 4)\}$$

and $\text{arc}_f(1) = \{2\}$, $\text{arc}_f(0) = \{3, 4\}$. This gives

g	$\times \circ \times = (\times \times)_1^2$	$\circ \times \circ \times = (\times \times)_0^3$	$\circ \times \times = (\times \times)_0^2$
$K(\frac{g}{f})$	1	2	$2z$

and $K(\frac{g}{f}) = 0$ for other values of g .

(3) Take $f = \times^2 \times \circ \circ \times$. One has $\text{arc}_f(0) = \{3, 6, 7, 8\}$ and

g	$\times \times \circ \circ \times \times = (f)_0^5$	$(f)_0^6$	$\times \times \circ \circ \times \circ \circ \times = (f)_0^7$	$(f)_4^5$
$K(\frac{g}{f})$	$z + z^2$	$1 + z$	2	1

Since $K(\frac{g}{f}) = 0$ implies $\lambda_1 > \nu_1 = 4$ we get $K(\frac{g}{f}) = 0$ for other values of g .

(4) Take $f = \times^2 \times \circ \circ \circ \times$. One has $\text{arc}_f(0) = \{3, 4, 7, 8\}$ which gives

$$K(\frac{(f)_0^6}{f}) = 2z, \quad K(\frac{(f)_0^7}{f}) = 2, \quad K(\frac{(f)_5^6}{f}) = 1$$

Since $K(\frac{g}{f}) = 0$ implies $\lambda_1 > \nu_1 = 5$ we get $K(\frac{g}{f}) = 0$ for other values of g .

3.5. Proof of Proposition 3.4

Theorem 4 in [30] gives (i) and establishes (ii), (iii) for $m = 1$ (in this case $\mathcal{B} = \{0\}$). From now on we assume that $m \geq 2$ and $\lambda \neq 0$. We set

$$\theta := \varepsilon_1 - \varepsilon_m.$$

3.5.1. Notation

Recall that $m = 2n + \ell$ and $n > 0$. For $\mu \in \mathcal{B}$ we write $\mu = (\mu_1, \dots, \mu_n)$ and set $\mu' := \mu|_{t_{(n-1)}}$, i.e., $\mu' = (\mu_2, \dots, \mu_n)$. We will denote the weight diagram of μ by $\text{diag}(\mu)$. For a polynomial $P \in \mathbb{Z}[z]$ we introduce $\overline{P} \in \{0, 1\}$ by $\overline{P} := P(0)$ modulo 2; we will also use the following notation: $(\sum_{i=-\infty}^{\infty} d_i z^i)_+ := \sum_{i=0}^{\infty} d_i z^i$.

3.5.2. Formulae from [30]

Theorem 4 in [30] can be written in the following form

$$K^{\frac{\theta}{2}, \nu} = 0 \text{ for } m = 2 \quad K^{\theta, \mu} = \delta_{0, \mu}(1 + z^{m-2}). \tag{18}$$

Theorem 3 in [30] gives for $m \geq 2$, $\lambda_1 > 1$ and $\nu \neq \lambda - \theta$

$$\begin{aligned} K^{\lambda, \lambda - \theta} &= 1, \\ K^{\lambda, \nu} &= (z^{-1} K^{\lambda - \theta, \nu})_+ && \text{for } \lambda_1 > \lambda_2 + 1, \text{ tail}(\nu) \leq \text{tail}(\lambda) \\ K^{\lambda, \nu} &= (z^{-1} K^{\lambda - \theta, \nu})_+ + \overline{K^{\lambda - \theta, \nu}} && \text{for } \lambda_1 > \lambda_2 + 1, \text{ tail}(\nu) > \text{tail}(\lambda) \\ K^{\lambda, \nu} &= 0 && \text{for } \lambda_1 = \lambda_2 + 1, \nu_1 \neq \lambda_2 \\ K^{\lambda, \nu} &= z K^{\lambda', \nu'} && \text{for } \lambda_1 = \lambda_2 + 1, \nu_1 = \lambda_2. \end{aligned} \tag{19}$$

3.5.3. Case $\lambda_1 \leq 1$

In this case $\lambda = 0, \theta$ or $\lambda = \frac{\theta}{2}$ for $m = 2$ (note that $\frac{\theta}{2} \notin P^+(\mathfrak{q}_m)$ for $m > 2$). For $m = 2$ there is no diagram f satisfying $(f)_a^b = \text{diag}(\frac{\theta}{2})$. If $(\text{diag}(\nu))_a^b = \text{diag}(\theta)$, then $a = 0$ and

$\nu = 0$, so $\text{arc}_{\text{diag}(\nu)}(0) = \{1, 2, \dots, m\}$. Comparing this with (18) we obtain (ii), (iii) for the case $\lambda_1 \leq 1$.

3.5.4. Case $n = 1$

In this case $\mathcal{B}_0 = \mathbb{N}\theta$ and $\mathcal{B}_{1/2} = (\mathbb{N} + \frac{1}{2})\theta$ for $\ell = 0$. The induction on λ_1 gives $K^{\lambda, \nu}(z) = \delta_{\lambda-\theta, \nu}$ (for $\lambda \neq 0$); this gives (ii), (iii) for the case $n = 1$.

3.5.5.

If $\nu = \lambda - \theta$, then $\text{diag}(\lambda) = (\text{diag}(\nu))_{\lambda_1-1}^{\lambda_1}$ and $K^{\lambda, \nu}(z) = 1$ by (19); thus (iii) holds for this case.

3.5.6.

Assume that $\nu \neq \lambda - \theta$. Set $j := \lambda_1 - \lambda_2 - 1$ and take $\mu := \lambda - j\theta$ (i.e., $\text{diag}(\mu)$ is obtained from $\text{diag}(\lambda)$ by moving the rightmost \times to the left “as much as possible”: for instance, if $\text{diag}(\lambda) = \times \circ \times \circ \circ \times$, then $\text{diag}(\mu) = \times \circ \times \times$). By (19), $K^{\lambda, \nu}(z) \neq 0$ implies $K^{\mu, \nu}(z) \neq 0$ which forces $\nu_1 = \mu_2$ (since $\mu_1 - 1 = \mu_2$). Hence $\nu_1 = \lambda_2$. We obtain

$$K^{\lambda, \nu}(z) \neq 0, \quad \nu \neq \lambda - \theta \implies \nu_1 = \lambda_2.$$

3.5.7.

We will prove (ii), (iii) by the induction on λ_1 (note that $\lambda_1 \geq \frac{1}{2}$ since $\lambda \neq 0$). The cases $\lambda_1 \leq 1$ and $n = 1$ are established above. From now till the end of the proof we assume

$$n \geq 2, \quad \lambda_1 > 1, \quad \nu_1 = \lambda_2, \quad \nu \neq \lambda - \theta.$$

Using $\nu_1 = \lambda_2$ we write $\text{diag}(\lambda), \text{diag}(\nu)$ in the form

$$\text{diag}(\lambda) = g * \times, \quad \text{diag}(\nu) = f * \circ \tag{20}$$

where the symbols $* \in \{\circ, \times\}$ occupy the position $\lambda_1 - 1$ in both diagrams (note that f, g do not have the same meaning as in 3.1). For example,

$$\begin{array}{ll} \text{diag}(\lambda) = & \circ \times \times \times \circ \times \times & \text{diag}(\nu) = & \times \times \times \times \circ \times \circ \\ * = \times, & g = \circ \times \times \times \circ & f = & \times \times \times \times \circ \end{array}$$

The formulae (19) give

$$\begin{aligned} K\left(\frac{f \circ \times}{f \times \circ}\right) &= 1 \\ K\left(\frac{g \circ \times}{f \circ \circ}\right) &= (z^{-1}K\left(\frac{g \times}{f \circ}\right))_+ \quad \text{if } \text{tail}(\nu) \leq \text{tail}(\lambda), \\ K\left(\frac{g \circ \times}{f \circ \circ}\right) &= (z^{-1}K\left(\frac{g \times}{f \circ}\right))_+ + \overline{K\left(\frac{g \times}{f \circ}\right)} \quad \text{if } \text{tail}(\nu) > \text{tail}(\lambda), \\ K\left(\frac{g \times \times}{f \times \circ}\right) &= zK\left(\frac{g \times}{f \circ}\right), \quad K\left(\frac{g \times \times}{f \circ \circ}\right) = 0. \end{aligned} \tag{21}$$

3.5.8. Proof of (ii)

Assume that $K^{\lambda,\nu}(z) \neq 0$.

If $* = \times$, then $K^{\lambda,\nu} = K(\frac{g \times \times}{f \times \circ}) = zK(\frac{g \times}{f \circ})$. By induction, $K(\frac{g \times}{f \circ}) \neq 0$ implies that $g \times = (f \circ)_a^{\lambda_1 - 1}$ for some a , which gives $g \times \times = (f \times \circ)_a^{\lambda_1}$.

If $* = \circ$, then $K^{\lambda,\nu} = K(\frac{g \circ \times}{f \circ \circ}) \neq 0$. By (21), this gives $K(\frac{g \times}{f \circ}) \neq 0$. By induction this implies $g \times = (f \circ)_a^{\lambda_1 - 1}$ for some a , which gives $g \circ \times = (f \circ \circ)_a^{\lambda_1}$.

This establishes (ii).

3.5.9.

The proof of (iii) occupies 3.5.9—3.5.11. We assume that $\text{diag}(\lambda) = (\text{diag}(\nu))_a^{\lambda_1}$ and $\nu \neq \lambda - \theta$. Then

$$\text{diag}(\lambda) = g * \times \quad \text{diag}(\nu) = f * \circ \quad g \times = (f \circ)_a^{\lambda_1 - 1}. \tag{22}$$

3.5.10. Case $a \neq 0$

In this case $\text{tail}(\nu) = \text{tail}(\lambda)$. Take $b' := \text{arc}_f(a)$ and $b := \text{arc}_{f*}(a)$.

If $* = \circ$, then $f = f*$ and $b = b'$. By induction we get

$$K(\frac{g \circ \times}{f \circ \circ}) = (z^{-1}K(\frac{g \times}{f \circ}))_+ = (z^{-1}(z^{b - (\lambda_1 - 1)}))_+ = (z^{b - \lambda_1})_+$$

as required. For $* = \times$ one has $\text{arc}(\lambda_1 - 1; \lambda_1) \in \text{Arc}(f \times \circ)$, so $b = b'$ if $b' < \lambda_1 - 1$ and $b = b' + 2$ otherwise. By induction we get

$$K(\frac{g \times \times}{f \times \circ}) = zK(\frac{g \times}{f \circ}) = z(z^{b' - (\lambda_1 - 1)})_+ = (z^{b - \lambda_1})_+.$$

This establishes the required formula for $a \neq 0$.

3.5.11. Case $a = 0$

In this case $\text{tail}(\nu) = \text{tail}(\lambda) + 1$. Set

$$i_- := \min A_{f*;\lambda_1}, \quad i_+ := \min A_{f*;\lambda_1}, \quad i'_- := \min A_{f;\lambda_1 - 1}, \quad i'_+ := \min A_{f;\lambda_1 - 1}$$

taking $i_{\pm} = -\infty$ (resp., $i'_{\pm} = -\infty$) if $A_{f*;\lambda_1} = \emptyset$ (resp., $A_{f;\lambda_1 - 1} = \emptyset$). By induction

$$K(\frac{g \times}{f \circ}) = z^{i'_- - (\lambda_1 - 1)} + z^{i'_+ - (\lambda_1 - 1)}.$$

If $* = \times$, then $i_{\pm} = i'_{\pm} + 2$ and (21) gives

$$K(\frac{g \times \times}{f \times \circ}) = zK(\frac{g \times}{f \circ}) = z^{i_- - \lambda_1} + z^{i_+ - \lambda_1}.$$

Consider the remaining case $* = \circ$. By (21) we have

$$K\left(\frac{g \circ \times}{f \circ \circ}\right) = (z^{-1}K\left(\frac{g \times}{f \circ}\right))_+ + \overline{K\left(\frac{g \times}{f \circ}\right)}.$$

Since the coordinates of \times in f are smaller than $\lambda_1 - 1$, $\text{arc}_f(0)$ contains all integers between i'_- and i'_+ . Thus $\lambda_1 - 1 \leq i'_- \leq i'_+$ and

$$A_{f^*, \lambda_1} = A_{f, \lambda_1} = \{i \mid i \neq \lambda_1 - 1, i'_- \leq i \leq i'_+\}.$$

If $i'_- \neq \lambda_1 - 1$, this gives $i_- = i'_-$ and $i_+ = i'_+$ which imply

$$K\left(\frac{g^* \times}{f^* \circ}\right) = z^{-1}K\left(\frac{g \times}{f \circ}\right) = z^{i_- - \lambda_1} + z^{i_+ - \lambda_1}.$$

If $i'_- = i'_+ = \lambda_1 - 1$, then $A_{f^*, \lambda_1} = \emptyset$ and thus $i_{\pm} = -\infty$. One has $K\left(\frac{g \times}{f \circ}\right) = 2$, so $K\left(\frac{g^* \times}{f^* \circ}\right) = 0 = z^{i_- - \lambda_1} + z^{i_+ - \lambda_1}$.

If $i'_- = \lambda_1 - 1 < i'_+$, then $i_- = \lambda_1$, $i_+ = i'_+$. In this case $K\left(\frac{g \times}{f \circ}\right) = 1 + z^{i'_+ - (\lambda_1 - 1)}$ and $K\left(\frac{g^* \times}{f^* \circ}\right) = 1 + z^{i_+ - \lambda_1}$.

We see that in all cases $K\left(\frac{g^* \times}{f^* \circ}\right) = z^{i_- - \lambda_1} + z^{i_+ - \lambda_1}$. This completes the proof of (iii). \square

4. The grading dex and the computation of $\text{ext}(\lambda; \nu)$

In this section we introduce the \mathbb{Z}_2 -grading dex and describe the graphs $G(\mathcal{B}; K^0)$. Then we describe the graphs $(\mathcal{C}; \text{ext})$ which were defined in Introduction.

4.1. The grading dex

Recall that $\ell = 1$ for $\mathfrak{osp}(2n + 2|2n)$, \mathfrak{q}_{2n+1} and $\ell = 0$ in other cases. For $\lambda, \nu \in \mathcal{B}$ we take $\lambda_1, \dots, \lambda_n$ as in 2.2 and introduce

$$\|\lambda\| := \begin{cases} \sum_{i=1}^n \lambda_i & \text{if } \mathfrak{g} \neq \mathfrak{osp}(2n + 2|2n) \\ \sum_{i=1}^n \lambda_i - \ell(n - \text{tail } \lambda) & \text{if } \mathfrak{g} = \mathfrak{osp}(2n + 2|2n) \end{cases} \tag{23}$$

$$\text{dex}(\lambda) := \|\lambda\| \pmod 2 \quad \text{dex}(\lambda; \nu) := \begin{cases} 0 & \text{if } \text{dex}(\lambda) = \text{dex}(\nu) \\ 1 & \text{if } \text{dex}(\lambda) \neq \text{dex}(\nu) \end{cases}$$

$$\text{tail}(\nu; \lambda) := \text{tail } \nu - \text{tail } \lambda.$$

Observe that $\|\tau(\lambda)\| = \|\lambda\|$ for $\mathfrak{g} = \mathfrak{osp}(2n + 2|2n)$.

4.1.1.

Corollary. Let $\lambda, \nu \in \mathcal{B}$ be such that $K^{\lambda, \nu}(z) \neq 0$.

- (i) For $\mathfrak{g} = \mathfrak{gl}(n|n)$ or $\mathfrak{g} = \mathfrak{q}_{2n}$ with $\mathcal{B} := \mathcal{B}_{1/2}$ one has $\lambda > \nu$ and $K^{\lambda, \nu}(z) = z^i$, for some i such that $i \equiv \text{dex}(\lambda; \nu) + 1$ modulo 2.
- (ii) Take $\mathfrak{g} = \mathfrak{osp}(2n + 2|2n), \mathfrak{osp}(2n + 1|2n)$ with $\lambda \neq 0$. Then $\lambda > \nu$, $\text{tail}(\nu; \lambda) \in \{0, 1, 2\}$, and $K^{\lambda, \nu}(z) = z^i$ for some i such that $i \equiv \text{dex}(\lambda; \nu) + 1$.
- (iii) Take $\mathfrak{g} = \mathfrak{osp}(2n|2n)$ with $\lambda \neq 0$. Then $\lambda > \nu$ and $\text{tail}(\nu; \lambda) \in \{0, 1, 2\}$. If $\text{tail}(\nu; \lambda) \neq 1$, then $K^{\lambda, \nu}(z) = z^i$; if $\text{tail}(\nu; \lambda) = 1$, then $K^{\lambda, \nu}(z)$ equals to z^i or to $z^i + z^j$ for some i, j such that $j < i$ and $j \equiv i$ modulo 2. In both cases i satisfies $i \equiv \text{dex}(\lambda; \nu) + 1$.
- (iv) Take $\mathfrak{g} = \mathfrak{q}_{2n+\ell}$ with $\lambda \neq 0$. Then $\lambda > \nu$ and $\text{tail}(\nu; \lambda) \in \{0, 1\}$.
 If $\text{tail}(\nu; \lambda) = 0$, then $K^{\lambda, \nu}(z) = z^i$ for some i such that $i \equiv \text{dex}(\lambda; \nu) + 1$.
 If $\text{tail}(\nu; \lambda) = 1$, then $K^{\lambda, \nu}(z) = z^i + z^j$ for some i, j such that $j \leq i$ and $i \equiv \text{dex}(\lambda; \nu) + 1 + \ell$.

Proof. By Proposition 3.3 (iv) we can assume $\mathfrak{g} \neq \mathfrak{osp}(2n + 1|2n)$. Theorems 3.2–3.4 immediately imply all assertions except $i \equiv \text{dex}(\lambda; \nu) + 1$ modulo 2 and $j \equiv i$ modulo 2 for $\mathfrak{osp}(2n|2n)$. We retain notation of 3.2–3.4. Recall that $K^{\lambda, \nu}(z) \neq 0$ implies $g = (f)_a^{\lambda_1}$ or $\mathfrak{g} = \mathfrak{osp}(2n + t|2n)$ and $g = (f)_{0,0}^{p, \lambda_1}$.

Consider the case $g = (f)_a^{\lambda_1}$. In this case

$$\text{dex}(\lambda; \nu) \equiv \begin{cases} \lambda_1 - a & \text{if } a \neq 0 \text{ or } \mathfrak{g} \neq \mathfrak{osp}(2n + 2|2n), \mathfrak{q}_{2n+1} \\ \lambda_1 - a + 1 & \text{if } a = 0 \text{ and } \mathfrak{g} = \mathfrak{osp}(2n + 2|2n), \mathfrak{q}_{2n+1}. \end{cases}$$

Consider the case when $\mathfrak{g} \neq \mathfrak{q}_{2n+\ell}$ or $a \neq 0$. In this case $i = b - \lambda_1$, where $b = \max \text{arc}_f(a)$. Observe that $b - a$ is odd except for the case when $\mathfrak{g} = \mathfrak{osp}(2n + 2|2n)$ and $a = 0$; in the latter case $b - a$ is even. Hence $i \equiv \text{dex}(\lambda; \nu) + 1$ if $\mathfrak{g} = \mathfrak{q}_{2n+\ell}$ or $a \neq 0$. For $\mathfrak{osp}(2n|2n)$ with $a = 0$ one has $j = b_- - \lambda_1$, where $b_- = \min \text{arc}_f(0)$ is odd; this gives $j \equiv \text{dex}(\lambda; \nu) + 1$.

Consider the case $\mathfrak{g} = \mathfrak{q}_{2n+\ell}$ with $a = 0$. One has $K^{\lambda, \nu}(z) = z^{i_- - \lambda_1} + z^{i_+ - \lambda_1}$, where $i_- \leq i_+ = \max\{s \in \text{arc}_f(0) \mid \lambda_1 \leq s < \max \text{arc}_f(0)\}$. Observe that $i_+ \equiv \ell + 1$, so $i_+ - \lambda_1 \equiv \text{dex}(\lambda; \nu) + 1 + \ell$ as required.

For the remaining case $\mathfrak{g} = \mathfrak{osp}(2n + t|2n)$ and $g = (f)_{0,0}^{p, \lambda_1}$ one has $\text{dex}(\lambda; \nu) \equiv p + \lambda_1$ modulo 2. In this case $i = q - \lambda_1$, where $\text{arc}(0; p, q)$ is a three-legged arch in $\text{Arc}(f)$. Since $q - p$ is odd, this implies $i \equiv \text{dex}(\lambda; \nu) + 1$. This completes the proof. \square

4.1.2. Remark

The coefficients of the character formulae obtained in [22], [35], [16] can be expressed in terms of the values $K^{\lambda, \nu}(-1)$. By above, if $K^{\lambda, \nu}(-1) \neq 0$, then

$$(-1)^{\text{dex}(\lambda;\nu)+1} K^{\lambda,\nu}(-1) = \begin{cases} 1 & \text{for } \mathfrak{gl}(n|n), \mathfrak{osp}(2n+1|2n), \mathfrak{osp}(2n+2|2n) \\ 1 & \text{for } \mathfrak{osp}(2n|2n), \mathfrak{q}_{2n+\ell} \text{ if } \text{tail}(\nu; \lambda) \neq 1 \\ 1 \text{ or } 2 & \text{for } \mathfrak{osp}(2n|2n) \text{ if } \text{tail}(\nu; \lambda) = 1 \\ (-2)^\ell & \text{for } \mathfrak{q}_{2n+\ell} \text{ if } \text{tail}(\nu; \lambda) = 1. \end{cases}$$

4.2. Example

For $n = 1$ the polynomials $K^{\lambda,\nu}$ can be presented by the following graphs where the arrows stand for $K^{\lambda,\nu} \neq 0$ and the solid arrows for $K^{\lambda,\nu}(0)$, so the solid arrows constitute the graph $G(\mathcal{B}; K^0)$. If $K^{\lambda,\nu}(z)$ is not a constant polynomial, we write $K^{\lambda,\nu}(z)$ near the corresponding arrow. Using Remark 2.2.4 we obtain

$$\mathfrak{gl}(1|1) : \quad \dots \longrightarrow -\beta \longrightarrow 0 \longrightarrow \beta \longrightarrow 2\beta \longrightarrow \dots$$

$$\mathfrak{osp}(2|2) : \quad \dots \longleftarrow \beta^\sigma \longleftarrow 0 \longrightarrow \beta \longrightarrow 2\beta \longrightarrow \dots$$

$$OSP(2|2) : \quad 0 \longrightarrow \beta \longrightarrow 2\beta \longrightarrow \dots$$

$$\mathfrak{osp}(4|2) : \quad \begin{array}{ccccccc} 0 & \longrightarrow & 2\beta & \longrightarrow & 3\beta & \longrightarrow & \dots \\ & \swarrow z & \uparrow & & & & \\ & & \beta & & & & \end{array}$$

$$\mathfrak{q}_2, \mathcal{B}_{1/2} : \quad 0 \longrightarrow \frac{\theta}{2} \longrightarrow \frac{3\theta}{2} \longrightarrow \frac{5\theta}{2} \longrightarrow \dots$$

$$\mathfrak{q}_2, \mathcal{B}_0 : \quad 0 \Longrightarrow \theta \longrightarrow 2\theta \longrightarrow 3\theta \longrightarrow \dots$$

$$\mathfrak{q}_3, \mathcal{B}_0 : \quad \begin{array}{ccccccc} 0 & \Longrightarrow & 2\theta & \longrightarrow & 3\theta & \longrightarrow & \dots \\ & \searrow 1+z & \uparrow & & & & \\ & & \theta & & & & \end{array}$$

For $\mathfrak{g} \neq \mathfrak{osp}(4|2)$ the grading dex is given by $\text{dex}(i\beta) \equiv i$, $\text{dex}(i\beta') \equiv i$, $\text{dex}(i\theta) \equiv i$; for $\mathfrak{osp}(4|2)$ one has $\text{dex}(i\beta) \equiv i - 1 + \delta_{i0}$.

4.3. Polynomials $\hat{K}^{\lambda, \nu}(z; w)$

Retain notation of 1.6, 1.8 and 2.5. Substituting \mathfrak{g} by $\mathfrak{g}_{(s)}$ we obtain the functors $\Gamma_{\bullet}^{\mathfrak{g}_{(s)}, \mathfrak{P}(s)}$ which satisfy the assumptions (A), (B) of 1.8.1. The formulae for $K_{(s)}^i(\lambda; \nu)$ can be obtained from the formulae for $K^i(\lambda; \nu)$ by changing λ_1 to λ_s and m to s in \mathfrak{q}_m -case. We set

$$\hat{K}^{\lambda, \nu}(z; w) := \sum_{i=1}^k \sum_{j=0}^{\infty} K_{(i)}^j(\lambda; \nu) z^i w^j. \tag{24}$$

Using 3.2–3.4 we obtain $\hat{K}^{\lambda, \lambda}(z, w) = 0$ for any $\lambda \in \mathcal{B}$ with $\text{tail } \lambda = 0$ (for $\mathfrak{gl}(n|n)$ this holds for any $\lambda \in \mathcal{B}$).

4.3.1.

Corollary. *Take $\lambda \in \mathcal{B}$.*

(i) *For $\mathfrak{g} = \mathfrak{gl}(n|n)$ and $\mathfrak{q}_{2n+\ell}$ one has*

$$\hat{K}^{\lambda, \nu}(z, w) \neq 0 \implies \lambda \geq \nu \ \& \ \nu \in \mathcal{B}.$$

(ii) *For $\mathfrak{osp}(2n + t|2n)$ one has*

$$\hat{K}^{\lambda, \nu}(z, w) \neq 0 \ \& \ \lambda \geq \nu \implies \nu \in \mathcal{B}.$$

Proof. Assume that $\hat{K}^{\lambda, \nu}(z; w) \neq 0$ for some $\nu \neq \lambda$; for $\mathfrak{osp}(2n + t|2n)$ we assume, in addition, $\lambda > \nu$.

Since $\hat{K}^{\lambda, \nu}(z; w) \neq 0$ one has $K_{(s)}^j(\lambda; \nu) \neq 0$ for some j, s . Set $\lambda' := \lambda|_{\mathfrak{t}_{(s)}}$, $\nu' := \nu|_{\mathfrak{t}_{(s)}}$ and let $\mathcal{B}' \subset P^+(\mathfrak{g}_{(s)})$ be the analogue of the set \mathcal{B} for $\mathfrak{g}_{(s)}$. Note that $\lambda' \in \mathcal{B}'$. By above,

(a) $K^j(\lambda'; \nu') = K_{(s)}^j(\lambda; \nu) \neq 0$

which implies

(b) $\nu' \in P^+(\mathfrak{g}_{(s)})$ and $\nu|_{\mathfrak{t}_{(s)}^+} = \lambda|_{\mathfrak{t}_{(s)}^+}$.

In particular, $\nu' \neq \lambda'$ (since $\nu \neq \lambda$ and $\nu|_{\mathfrak{t}_{(s)}^+} = \lambda|_{\mathfrak{t}_{(s)}^+}$). In the \mathfrak{osp} -case combining (b) and $\nu < \lambda$ we obtain $\nu' < \lambda'$; since 0 is the minimal element in $P^+(\mathfrak{osp}(2s + t|2s))$ this implies $\lambda' \neq 0$. We conclude that $K^j(\lambda'; \nu')$ is given by 3.2– 3.4 (since $\lambda' \neq 0$ for the \mathfrak{osp} -case). Using 3.2– 3.4 we deduce from (a)

(c) $\nu' \in \mathcal{B}'$ and $\nu' < \lambda'$

for all cases. Combining $\nu' < \lambda'$ with (b) we obtain $\lambda > \nu$ for $\mathfrak{gl}(n|n)$ and \mathfrak{q}_{2n+l} .

Let us show that $\nu \in \mathcal{B}$. Combining (b) and (c) we conclude that $\nu + \rho$ can be written in the form appeared in 2.2. Moreover 3.2– 3.4 give

$$(d) \nu_{n+1-s} < \lambda_{n+1-s}$$

Combining (b) and (c) we conclude that ν_i s are integral (resp., non-negative integral, in $\mathbb{N} + 1/2$) for $\mathfrak{g} = \mathfrak{gl}(n|n)$ (resp., for \mathcal{B}_0 with $\mathfrak{g} \neq \mathfrak{gl}(n|n)$, for $\mathcal{B}_{1/2}$). By (b)

$$\nu_i = \lambda_i \quad \text{for } 1 \leq i \leq n - s. \tag{25}$$

Since $\lambda \in \mathcal{B}$ one has $\lambda_{n+1-s} \leq \lambda_{n-s} = \nu_{n-s}$; using (d) we get

$$\nu_{n+1-s} < \nu_{n-s}. \tag{26}$$

For $\mathfrak{gl}(n|n)$ -case and for \mathfrak{q}_{2n} with $\mathcal{B}_{1/2}$ combining (c), (25), (26) and the condition $\lambda \in \mathcal{B}$ we get $\nu_i < \nu_{i+1}$ for each i . For other cases we get either $\nu_i \leq \nu_{i+1}$ or $\nu_i = \nu_{i+1} = 0$ for each i . This implies $\nu \in \mathcal{B}$. \square

4.3.2. Example

The following example shows that $\hat{K}^{\lambda,\nu}(z, w) \neq 0$ does not imply $\lambda \geq \nu$ or $\nu \in \mathcal{B}$ in \mathfrak{osp} -case. By [13], $K^{0,\varepsilon_1}(z) = z$ for $\mathfrak{osp}(3|2)$; this implies $\hat{K}^{0,\varepsilon_2} = zw$ for $\mathfrak{osp}(5|4)$ whereas $0 < \varepsilon_2$ and $\varepsilon_2 \notin P^+(\mathfrak{osp}(5|4))$ (and so $\varepsilon_2 \notin \mathcal{B}$).

4.3.3.

Corollary. Take $\lambda \neq \nu \in \mathcal{B}$ and set $s := n + 1 - \max\{i \mid \lambda_i \neq \nu_i\}$.

(i) Take $\mathfrak{g} = \mathfrak{gl}(n|n), \mathfrak{q}_{2n+l}$. If $\hat{K}^{\lambda,\nu}(z; w) \neq 0$, then

$$\hat{K}^{\lambda,\nu}(z; w) = \begin{cases} z^i w^s & \text{for } \mathfrak{gl}(n|n) \\ z^i w^s & \text{for } \mathfrak{q}_{2n+l} \quad \text{if } \text{tail } \lambda = \text{tail } \nu \\ (z^i + z^j)w^s & \text{for } \mathfrak{q}_{2n+l} \quad \text{if } \text{tail } \lambda \neq \text{tail } \nu \end{cases}$$

with $0 \leq j \leq i$ in the last case.

(ii) Take $\mathfrak{g} = \mathfrak{osp}(2n + t|2n)$. If $\hat{K}^{\lambda,\nu}(z; w) \neq 0$ and $\text{tail}(\lambda) \leq \text{tail}(\nu)$, then

$$\hat{K}^{\lambda,\nu}(z; w) = \begin{cases} z^i w^s & \text{for } \mathfrak{osp}(2n + 1|2n), \mathfrak{osp}(2n + 2|2n) \\ z^i w^s & \text{for } \mathfrak{osp}(2n|2n) \quad \text{if } \text{tail } \lambda = \text{tail } \nu \\ z^i w^s \text{ or } (z^i + z^{i-2i'})w^s & \text{for } \mathfrak{osp}(2n|2n) \quad \text{if } \text{tail } \lambda \neq \text{tail } \nu \end{cases}$$

with $0 \leq i - 2i' < i$ in the last case.

In all cases $i \equiv \text{dex}(\lambda) - \text{dex}(\nu) + 1$ modulo 2.

Proof. The formulae in 3.2, 3.4 give (i). For (ii) take λ', ν' as in the proof of Corollary 4.3.1. The conditions $\lambda \neq \nu$ and $\text{tail}(\lambda) \leq \text{tail}(\nu)$ imply $\lambda' \neq \nu'$ and $\text{tail}(\lambda') \leq \text{tail}(\nu')$ which force $\lambda' \neq 0$. Therefore $K_{(s)}^i(\lambda; \nu) = K^i(\lambda'; \nu')$ is given by 3.3; this gives (ii). \square

4.4. Graph $G(\mathcal{B}; K^0)$

Retain notation of 1.9.1. By 4.3.1, if $\nu \rightarrow \lambda$ is an edge in $G(\mathfrak{t}^*, K^0)$ with $\lambda \in \mathcal{B}$, then $\nu \in \mathcal{B}$. In other words, $B(\lambda) \subset \mathcal{B}$ for each $\lambda \in \mathcal{B}$. Using 3.2–3.4 we obtain the following description for $G(\mathcal{B}; K^0)$.

4.4.1. Case $\mathfrak{gl}(n|n)$

In this case $\nu \mapsto \lambda$ is an edge in $G(\mathcal{B}; K^0)$ if and only if the diagram of λ is obtained from the diagram of ν by moving one symbol \times along the arch originated at this symbol. Each vertex has exactly n direct successors.

4.4.2. Case \mathfrak{q}_{2n} with $\mathcal{B} = \mathcal{B}_{1/2}$

In this case $\nu \mapsto \lambda$ is an edge in $G(\mathcal{B}; K^0)$ if and only if the diagram of λ is obtained from the diagram of ν by moving one symbol \times along the arch originated at this symbol. Each vertex has exactly n direct successors.

4.4.3. Case $\mathfrak{osp}(2n+t|2n)$

The map τ gives an isomorphism between the graphs $G(\mathcal{B}; K^0)$ for $t = 1$ and $t = 2$. For $t = 0, 2$ an edge $\nu \mapsto \lambda$ appears in $G(\mathcal{B}; K^0)$ if and only if the diagram of λ is obtained from the diagram of ν by one of the following operations:

- moving one symbol \times from the zero position to the farthest position connected to the zero position;
- moving one symbol \times along the two-legged arch originated at this symbol

and, for $t = 0$, the diagrams of λ and ν do not have different signs (for $t = 2$ the diagrams do not have signs). As a result, for $t = 1, 2$ each vertex has exactly n direct successors; for $t = 0$ this holds for the vertices ν with $\text{tail } \nu = 0$ (observe that for $n = 1$ the vertex 0 has two direct successors $\delta_1 \pm \varepsilon_1$).

4.4.4. Case \mathfrak{q}_{2n+l} with $\mathcal{B} = \mathcal{B}_0$

Let $\text{arc}(0; b', b)$ be the maximal three-legged arch in $\text{Arc}(\nu)$. By 3.4, $\nu \mapsto \lambda$ is an edge in $G(\mathcal{B}; K^0)$ if and only if the diagram of λ is obtained from the diagram of ν by moving one symbol \times from a position a to a free position a' connected with a subject to the condition $a' \neq b$; the edge $\nu \mapsto \lambda$ is simple if $a' \neq b'$ and is double if $a' = b'$. Note that the number of three-legged arches in $\text{Arc}(\nu)$ is equal to $\text{tail } \nu$. We conclude that each

vertex ν is the origin of $n + \text{tail } \nu$ edges with no double edges if $\text{tail } \nu = 0$ and a unique double edge if $\text{tail } \nu > 0$.

4.4.5.

Corollary.

- (i) For the cases $\mathfrak{g} = \mathfrak{gl}(n|n), \mathfrak{osp}(2n+t|2n)$ with $\mathcal{B} = \mathcal{B}_0$ and for $\mathfrak{g} = \mathfrak{q}_{2n}$ with $\mathcal{B} = \mathcal{B}_{1/2}$ the map dex gives a bipartition of $G(\mathcal{B}; K^0)$.
- (ii) The graph does not have multiedges except for the case $(\mathfrak{q}_{2n+\ell}, \mathcal{B}_0)$ where the double edges appear.

4.4.6.

Fix $p \in \mathbb{Z}$ for $\mathfrak{g} = \mathfrak{gl}(n|n)$, $p \in \mathbb{N} - 1/2$ for $\mathcal{B}_{1/2}$ and $p \in \mathbb{N}$ for other cases. We set

$$\mathcal{B}_{>p} = \{\lambda \in \mathcal{B} \mid \lambda_n > p\}, \quad B_+ := \{\mu \in \mathbb{N}^n \mid \mu_1 > \mu_2 > \dots > \mu_n > 0\}$$

and identify $\mathcal{B}_{>p}$ with B_+ via the map $\mu \mapsto (\mu_1 - p; \mu_2 - p; \dots; \mu_n - p)$. Note that the weight diagram of $\mu \in B_+$ contains \circ or \times in each position and the corresponding arch diagram “does not depend on the type of \mathfrak{g} ”. By 3.2–3.4 for $\nu, \lambda \in \mathcal{B}_{>p}$ the polynomials $\hat{K}^{\lambda, \nu}(z; w)$ are the same for all types of \mathfrak{g} and \mathcal{B} (for fixed n). In particular, the induced subgraphs $(\mathcal{B}_{>p}; K^0)$ are isomorphic for all types of \mathfrak{g} and \mathcal{B} .

Notice that $\mathcal{B}_{>-1/2} = \mathcal{B}_{1/2}$ so for each p the graph $(\mathcal{B}_{>p}; K^0)$ is isomorphic to $(\mathcal{B}_{1/2}; K^0)$.

4.5. Graph $(\mathcal{B}; \text{ext})$

Recall that $\text{ext}(\lambda; \nu) = \text{ext}(\nu; \lambda)$ and $\text{ext}(\lambda; \nu) = 0$ if $\lambda \in \mathcal{B}, \nu \notin \mathcal{B}$. Retain notation of 1.9. One has

$$s(\lambda; \nu) = n + 1 - \max\{i \mid \lambda_i = \nu_i\}.$$

By Corollary 4.3.3, each pair $(\lambda; \nu)$ with $\lambda \neq \nu$ is K^i -stable for any i .

The following corollary describes the graph $(\mathcal{B}; \text{ext})$ for $(\mathfrak{g}, \mathcal{B}) \neq (\mathfrak{q}_{2n+\ell}, \mathcal{B}_0)$ and gives some information for the case $(\mathfrak{q}_{2n+\ell}, \mathcal{B}_0)$.

4.5.1.

Corollary. Take $\lambda \in \mathcal{B}$ and $\nu \in \mathcal{B}$ with $\nu < \lambda$.

- (i) If $(\mathfrak{g}, \mathcal{B}) \neq (\mathfrak{q}_{2n+\ell}, \mathcal{B}_0)$, then $\text{ext}(\lambda; \nu) = k_0(\lambda; \nu) \leq 1$. The module $\Gamma^{\mathfrak{q}, p} L_p(\lambda)$ has a semisimple radical.
- (ii) If $(\mathfrak{g}, \mathcal{B}) \neq (\mathfrak{q}_{2n+\ell}, \mathcal{B}_0)$, then dex is a bipartition of the graph $(\mathcal{B}; \text{ext})$.

- (iii) If $(\mathfrak{g}, \mathcal{B}) = (\mathfrak{q}_{2n+\ell}, \mathcal{B}_0)$, then $\text{ext}(\lambda; \nu) \leq k_0(\lambda; \nu)$. If $\lambda_n > 1 + \ell$, then
 - $\text{ext}(\lambda; \nu) = k_0(\lambda; \nu) \leq 1$;
 - $k_0(\lambda; \nu) \neq 0$ implies $\text{dex}(\nu) \neq \text{dex}(\lambda)$;
 - the module $\Gamma^{\mathfrak{g}, \mathfrak{p}} L_{\mathfrak{p}}(\lambda)$ as a semisimple radical.

Proof. For $\mathfrak{g} \neq \mathfrak{q}_{2n+\ell}$ one has $\mathfrak{h} = \mathfrak{t}$ and $\text{ext}(\lambda; \lambda) = 0$; for \mathfrak{q}_{2n} one has $\text{ext}(\lambda; \lambda) = 0$ for any $\lambda \in \mathcal{B}_{1/2}$. Combining Corollaries 1.9.3 and 4.4.5 we obtain (i), (ii) and the inequality $\text{ext}(\lambda; \nu) \leq k_0(\lambda; \nu)$ in (iii). Take $(\mathfrak{g}, \mathcal{B}) = (\mathfrak{q}_{2n+\ell}, \mathcal{B}_0)$. The assumption $\lambda_n > 1 + \ell$ gives $B(\lambda) \subset \mathcal{B}_{>0}$ (see 4.4.6 for notation). By 4.4.6 the map dex is a bipartition of the graph $G(B(\lambda), K^0)$ and $k_0(\lambda; \nu) = 1$ for each $\nu \in B(\lambda)$. Using Corollary 1.9.3 we obtain all assertions of (iii). \square

4.5.2. Example

For $n = 1$ the graphs $G(\mathcal{B}; K^0)$ are given in 4.2. The corresponding ext-graphs, A_{∞} for $\mathfrak{gl}(1|1)$, $\mathfrak{osp}(2|2)$, D_{∞} for $\mathfrak{osp}(4|2)$ and A_{∞} for the rest of the cases, appear in Introduction. In agreement with Corollary 4.5.1 (i) for $(\mathfrak{g}, \mathcal{B}) \neq (\mathfrak{q}_m, \mathcal{B}_0)$ the ext-graph can be obtained from $G(\mathcal{B}; K^0)$ by erasing the dotted arrows and changing \longrightarrow to \longleftrightarrow ; in this case dex is the bipartition of the ext-graph. In the remaining cases (for $n = 1$) the ext-graphs are

$$\begin{aligned} \mathfrak{q}_2, \mathcal{B}_0 : \quad & 0 \longleftrightarrow \theta \longleftrightarrow 2\theta \longleftrightarrow 3\theta \longrightarrow \dots \\ \mathfrak{q}_3, \mathcal{B}_0 : \quad & \theta \longleftrightarrow 0 \longleftrightarrow 2\theta \longleftrightarrow 3\theta \longrightarrow \dots \end{aligned}$$

see [27], [21]. Combining 4.2 and 4.5.2, we conclude that for \mathfrak{q}_2 the radical of $\Gamma^{\mathfrak{g}, \mathfrak{p}} L_{\mathfrak{p}}(\theta)$ is an indecomposable isotypical module of length two with the cosocle isomorphic to $L_{\mathfrak{g}}(0)$, and for \mathfrak{q}_3 the radical of $\Gamma^{\mathfrak{g}, \mathfrak{p}} L_{\mathfrak{p}}(2\theta)$ is a module of length three with the subquotients isomorphic to $L_{\mathfrak{g}}(0), L_{\mathfrak{g}}(0), L_{\mathfrak{g}}(\theta)$ and the cosocle isomorphic to $L_{\mathfrak{g}}(0)$.

4.5.3. Remark

Take $\mathfrak{g} = \mathfrak{q}_{2n+1}$ and $\lambda \in \mathcal{B}_0$ such that $\lambda_n = 1$ and $\lambda_{n-1} > 4$. Let us show that conclusions of Corollary 4.5.1 (iii) hold for such λ . Take μ such that $k_0(\lambda; \mu) \neq 0$. Since $\text{diag } \lambda = \Rightarrow \times \circ \circ \circ g$ for some diagram g one has $\text{diag } \mu = \overset{\times}{\Rightarrow} \circ \circ \circ \circ g$ or $\text{diag } \mu = \Rightarrow \times \circ \circ \circ f$ with $g = (f)_a^b$. In both cases $k_0(\lambda; \mu) = 1$ and $\text{dex}(\mu) \neq \text{dex}(\lambda)$. Hence $G(B(\lambda); K^0)$ is bipartite, $\Gamma^{\mathfrak{g}, \mathfrak{p}} L_{\mathfrak{p}}(\lambda)$ has a semisimple radical and Corollary 1.9.3 gives $\text{ext}(\lambda; \nu) = k_0(\lambda; \nu)$.

4.5.4. Remark

Fix $p \in \mathbb{Z}$ for $\mathfrak{g} = \mathfrak{gl}(n|n)$, $p \in \mathbb{N} - 1/2$ for $\mathcal{B}_{1/2}$ and $p \in \mathbb{N}$ for $\mathfrak{osp}(2n + t|2n)$, $p \in \mathbb{N}_{>\ell}$ for $(\mathfrak{q}_{2n+\ell}, \mathcal{B}_0)$. Retain notation of 4.4.6. By Corollary 4.5.1 $\text{ext}(\lambda; \nu) = k_0(\lambda; \nu)$ for each $\lambda, \nu \in \mathcal{B}_{>p}$ with $\lambda > \nu$. In the light of 4.4.6, for $\lambda \neq \nu$ the value $\text{ext}(\lambda; \nu)$ does not depend on \mathfrak{g} and p (under the identification of $\mathcal{B}_{>p}$ with B_+). Let \mathcal{C}_+ be the Serre

subcategory⁵ of $\mathcal{F}in(\mathfrak{g})$ generated by $L(\lambda)$ with $\lambda \in \mathcal{B}_{>p}$. By above, the graphs $(\mathcal{C}_+; \text{ext})$ are naturally isomorphic for all \mathfrak{g} with p as above. For $\mathfrak{gl}(n|n)$ and $\mathfrak{osp}(2n + t|2n)$ this implies the isomorphisms between Ext^1 -graphs of \mathcal{C}_+ (in these cases Ext^1 -graphs of \mathcal{C}_+ have two connected components which differ by Π).

4.6. Proof of Theorem A

Let $\tilde{\mathfrak{g}}$ be one of the algebras $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(M|2n)$ or \mathfrak{q}_m . We will say that weights $\lambda, \nu \in P^+(\tilde{\mathfrak{g}})$ have the same central character if $L(\lambda), L(\nu)$ have the same central character. The computation of $\text{ext}(\lambda; \nu)$ for arbitrary $\lambda, \nu \in P^+(\tilde{\mathfrak{g}})$ can be reduced to the case $\lambda, \nu \in \mathcal{B}$ with the help of translation functors which map a simple module in a given block to an isotypical semisimple module in another block of the same atypicality. For $\tilde{\mathfrak{g}} \neq \mathfrak{q}_m$, these semisimple modules are simple and, by [22], each block of atypicality k in $\mathcal{F}in(\mathfrak{gl}(m|n))$ (resp., in $\mathcal{F}in(\mathfrak{osp}(M|2n))$) is equivalent to the principal block in $\mathfrak{gl}(k|k)$ (resp., in $\mathfrak{osp}(2k + t|2k)$). In particular, if $L(\mu), L(\nu)$ have the same central character, then $\text{ext}(\mu; \nu) = \text{ext}(\bar{\mu}; \bar{\nu})$, where $\bar{\mu}, \bar{\nu}$ are the corresponding weights in \mathcal{B} ($\bar{\nu}$ is described in [22], Section 6). This gives Theorem A for $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(M|2n)$ and describes the graph $(\mathcal{F}in(\mathfrak{g}); \text{ext})$ in these cases.

For \mathfrak{q}_m -case the situation is more complicated, see [33].

4.6.1. Weight diagrams for \mathfrak{q}_m

For $\mu = \sum_{i=1}^m a_i \varepsilon_i$ denote by $\text{core}(\mu)$ the set obtained from $\{a_i\}_{i=1}^m$ by deleting the maximal number of pairs satisfying $a_i + a_j = 0$; for example, for $m = 8$ $\text{core}(2\varepsilon_1 + \varepsilon_2 - \varepsilon_8) = \{2; 0\}$. From the description of the center of $\mathcal{U}(\mathfrak{q}_m)$ obtained in [34], it follows that $L(\lambda), L(\mu)$ have the same central character if and only if $\text{core}(\lambda) = \text{core}(\mu)$. We set $\ell(\lambda) := 0$ if $0 \notin \text{core}(\lambda)$ and $\ell(\lambda) := 1$ otherwise.

The weight diagram for $\mu = \sum_{i=1}^m a_i \varepsilon_i \in P^+(\mathfrak{q}_M)$ is constructed by the following procedure: we put $>$ (resp., $<$) to the p th position if $a_i = p$ (resp., $a_i = -p$) for some i , add \circ to all empty positions and then glue each pair $>, <$ and each pair $>, >$ (which could occur only at the zero position) to one symbol \times . For instance

$$\begin{array}{lll} \mu = \varepsilon_1 - \varepsilon_3 - 3\varepsilon_4 & \nu = 3\varepsilon_1 - \varepsilon_4 & \lambda = 4\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3 - 2\varepsilon_4 \\ \text{diag } \mu = > \times \circ < & \text{diag } \nu = \times < \circ > & \text{diag } \lambda = \circ < \times > . \end{array}$$

If $\mu \in \mathcal{B}$ the resulting diagram coincides with the diagram constructed in 2.2.1.

The symbols $>, <$ are called *core symbols*. By above, λ, μ have the same central character if and only if all core symbols in their diagrams occupy the same positions (for instance, in the above example ν and λ have the same central character).

In this paper we define the atypicality of the weight to be the number of \times in the diagram. By contrast, in [20] the atypicality of the weight is defined as the number of

⁵ by Serre subcategory generated by a set of simple modules we mean the full subcategory consisting of the modules of finite length whose all simple subquotients lie in a given set.

\times in the diagram if the diagram does not have $>$ at the zero position and is equal to the number of \times plus $\frac{1}{2}$ if the diagram has $>$ at the zero position. In [20] the symbol $>$ at the zero position is not considered as a core symbol; note that for this definition it is still true that λ, μ have the same central character if and only if all core symbols in their diagrams occupy the same positions.

Let $\eta \in P^+(\mathfrak{q}_M)$ be a weight of atypicality $n > 0$. We denote by $\bar{\eta}$ the weight in $P^+(\mathfrak{q}_{2n+\ell})$ with the weight diagram which is obtained from $\text{diag } \eta$ by erasing all core symbols at the non-zero positions. For the above example we have

$$\begin{array}{lll} \text{diag } \bar{\mu} = > \times & \text{diag } \bar{\nu} = \times & \text{diag } \bar{\lambda} = \circ \times \\ \bar{\mu} = \varepsilon_1 - \varepsilon_3 & \bar{\nu} = 0 & \bar{\lambda} = \varepsilon_1 - \varepsilon_2. \end{array}$$

Note that $\bar{\eta} \in \mathcal{B}_0$ if η is integral and $\bar{\eta} \in \mathcal{B}_{1/2}$ if η is half-integral (for example, the weight $\eta = \frac{3}{2}\varepsilon_1 - \frac{1}{2}\varepsilon_2 - \frac{3}{2}\varepsilon_3$ has the diagram $> \times$, so $\text{diag } \bar{\eta} = \times$ and $\bar{\eta} = \frac{1}{2}\varepsilon_1 - \frac{1}{2}\varepsilon_2$).

4.6.2.

Proposition. *For any $\eta, \zeta \in P^+(\mathfrak{q}_m)$ one has $\text{ext}(\eta; \zeta) = \text{ext}(\bar{\eta}; \bar{\zeta})$ if η, ζ have the same central character.*

4.6.3. Outline of the proof

Take $\tilde{\mathfrak{g}} = \mathfrak{q}_m$ with a triangular decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_+ \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_-$. A weight $\mu \in P^+(\tilde{\mathfrak{g}})$ is called *stable* if all symbols \times precede all core symbols with non-zero coordinates (in the above example μ, ν are stable weights and λ is not stable).

Let η be a stable weight of atypicality n . Then $\tilde{\mathfrak{g}}$ contains a subalgebra $\mathfrak{g} \cong \mathfrak{q}_{2n+\ell(\eta)}$ with a compatible triangular decomposition such that the restriction of η to the Cartan subalgebra of \mathfrak{g}_0 equal to $\bar{\eta}$ (in the above example, for μ one has $\mathfrak{g} \cong \mathfrak{q}_3$ corresponding to $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and for ν one has $\mathfrak{g} \cong \mathfrak{q}_2$ corresponding to $\varepsilon_2, \varepsilon_3$). By [31], Corollary 1 for $\mathfrak{p} := \mathfrak{g} + \tilde{\mathfrak{b}}$ one has $\Gamma_0^{\tilde{\mathfrak{g}}, \mathfrak{p}} L_{\mathfrak{p}}(\eta) = L_{\tilde{\mathfrak{g}}}(\eta)$ and $\Gamma_i^{\tilde{\mathfrak{g}}, \mathfrak{p}} L_{\mathfrak{p}}(\eta) = 0$ for $i > 0$. Combining 1.5, 1.7.3 and 1.4.3 we obtain $\text{ext}(\eta; \zeta) = \text{ext}(\bar{\eta}; \bar{\zeta})$ if η, ζ are stable weights with the same central character.

The general case can be reduced to the stable case with the help of translation functors described in [2]. A translation functor which preserves the degree of atypicality and the value of $\ell(\eta)$ transforms $L(\eta)$ to $L(\eta') \oplus \Pi L(\eta')$, where $\text{diag } \eta'$ is obtained from $\text{diag } \eta$ by permuting two neighboring symbols at non-zero positions if exactly one of these symbols is a core symbol: for instance, for λ as above we can obtain λ' s with the diagrams $\circ \langle \rangle \times$ and $\circ \times \langle \rangle$ (the last diagram is stable). Note that $\bar{\eta} = \bar{\eta}'$. Using these functors we can transform any two simple modules $L(\eta), L(\zeta)$ with the same central character to the modules $L(\eta')^{\oplus r} \oplus \Pi L(\eta')^{\oplus r}, L(\zeta')^{\oplus r} \oplus \Pi L(\zeta')^{\oplus r}$, where η', ζ' are stable weights with the same central character and $\bar{\eta} = \bar{\eta}', \bar{\zeta} = \bar{\zeta}'$ (the diagrams of η' and ζ' are stable diagrams obtained from the diagrams of η and ζ by moving all core symbols from the non-zero position “far enough” to the right). It is not hard to show that these

functors map a module from $\mathcal{N}(\eta; \zeta; m)$ to a module of the form $M \oplus \Pi M$, where M is a direct sum of r modules from $\mathcal{N}(\eta', \zeta'; m)$. This gives $\text{ext}(\eta; \zeta) \leq \text{ext}(\eta'; \zeta')$. Using the same set of functors we can transform $L(\eta'), L(\zeta')$ to the modules $L(\eta)^{\oplus r} \oplus \Pi L(\eta)^{\oplus r}$, $L(\zeta)^{\oplus r} \oplus \Pi L(\zeta)^{\oplus r}$; this implies $\text{ext}(\eta'; \zeta') \leq \text{ext}(\eta; \zeta)$. Since η', ζ' are stable we obtain $\text{ext}(\eta; \zeta) = \text{ext}(\eta'; \zeta') = \text{ext}(\bar{\eta}; \bar{\zeta})$ as required. \square

4.6.4.

The arch diagrams for an arbitrary $\lambda \in P^+(\mathfrak{q}_M)$ are constructed in the same way as the arch diagrams for $\lambda \in \mathcal{B}$: starting from the rightmost symbol \times in the weight diagram $\text{diag}(\lambda)$ we connect each symbol \times at the non-zero position with the next free symbol \circ , then each symbol \times at the zero position with the next two free symbols \circ and then add wobbly arch if there is $>$ at the zero position. There is a natural bijection between the arches in $\text{Arc}(\lambda)$ and $\text{Arc}(\bar{\lambda})$.

4.6.5.

Corollary. For $\lambda > \nu \in P^+(\mathfrak{q}_m)$ one has

- (i) $\text{ext}(\lambda; \nu) \leq 2$;
- (ii) if $\text{ext}(\lambda; \nu) \neq 0$, then $\text{diag } \lambda$ can be obtained from $\text{diag } \nu$ by moving one symbol \times along the arch in $\text{Arc}(\nu)$;
- (iii) if $\text{diag } \bar{\lambda}$ does not have \times at the position $0, 1, 1 + \ell(\lambda)$, then $\text{ext}(\lambda; \nu) = 1$ if $\text{diag } \lambda$ can be obtained from $\text{diag } \nu$ by moving one symbol \times along the arch and $\text{ext}(\lambda; \nu) = 0$ otherwise.

4.6.6.

The above Corollary implies Theorem A for \mathfrak{q}_m and gives a description of the graph $(\mathcal{F}in(\mathfrak{q}_m)_{1/2}, \text{exp})$, where $\mathcal{F}in(\mathfrak{q}_m)_{1/2}$ is the full subcategory consisting of the modules with half-integral weights. Note that $\text{dex}(\bar{\lambda})$ is a bipartition of this graph.

By above, $\text{ext}_{\mathfrak{q}_3}(2\theta, \theta) = 0$, so the converse of (ii) does not hold (in this case $\text{diag } 2\theta = > \times$, so \times occurs at the position $1 + \ell(2\theta) = 2$).

4.7. Properties (Dex1), (Dex2)

Consider the case $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(M|2n)$. The map $\text{dex}(\bar{\lambda})$ is a bipartition of the graph $(\mathcal{F}in(\mathfrak{g}); \text{exp})$. The map $\text{Irr}(\mathcal{F}in(\mathfrak{g})) \rightarrow \mathbb{Z}_2$ given by $L(\lambda), \Pi L(\lambda) \mapsto \text{dex}(\bar{\lambda})$ satisfies (Dex1), but does not satisfy (Dex2). A map satisfying (Dex1) and (Dex2) can be constructed using a certain decomposition $\mathcal{F}in(\mathfrak{g}) = \mathcal{F} \oplus \Pi \mathcal{F}$ (for atypical modules N we take $N \in \mathcal{F}$ if $N_{\bar{i}} = \sum_{\mu: p(\mu)=\bar{i}} N_{\mu}$, where $p(\mu)$ is given by $p(\varepsilon_i) = \bar{0}, p(\delta_j) = \bar{1}$). Taking $\text{dex}(L(\lambda)) := \text{dex}(\bar{\lambda})$ for $L(\lambda) \in \mathcal{F}$ and $\text{dex}(L(\lambda)) := \text{dex}(\bar{\lambda}) + \bar{1}$ for $L(\lambda) \in \Pi \mathcal{F}$ we obtain a map satisfying (Dex1) and (Dex2), see [24], [15] for details.

For the case $(\mathfrak{q}; \frac{1}{2})$ the map $\text{dex}(\bar{\lambda})$ is a bipartition of the graph $(\mathcal{F}in(\mathfrak{q}_m)_{1/2}; \text{exp})$ (where $\mathcal{F}in(\mathfrak{g})_{1/2}$ is the full subcategory of $\mathcal{F}in(\mathfrak{q}_m)$ consisting of the modules with half-integral weights); the map $\text{Irr}(\mathcal{F}in(\mathfrak{q}_m)_{1/2}) \rightarrow \mathbb{Z}_2$ given by $L(\lambda), \Pi L(\lambda) \mapsto \text{dex}(\bar{\lambda})$ satisfies (Dex1).

In the remaining case $(\mathfrak{q}; \mathcal{C})$ we have $\mathfrak{g} := \mathfrak{q}_m$ and \mathcal{C} is the Serre subcategory generated by $L(\lambda), \Pi L(\lambda)$ with $\bar{\lambda}$ satisfying the assumption of Corollary 4.6.5 (iii). By above, $\text{dex}(\bar{\lambda})$ is a bipartition of the graph $(\mathcal{C}; \text{exp})$ and the map $\text{Irr}(\mathcal{C}) \rightarrow \mathbb{Z}_2$ given by $L(\lambda), \Pi L(\lambda) \mapsto \text{dex}(\bar{\lambda})$ satisfies (Dex1).

4.8. Remark

If \mathcal{C} is a full Π -invariant subcategory of $\mathcal{O}(\mathfrak{g})$ and \mathfrak{g} is a Kac-Moody superalgebra, then the Ext^1 -graph of \mathcal{C} is a disjoint union of two copies of the graph $(\mathcal{C}; \text{ext})$. In particular, if ext-graphs of $\mathcal{C} \subset \mathcal{O}(\mathfrak{g})$ and $\mathcal{C}' \subset \mathcal{O}(\mathfrak{g}')$ are isomorphic, then Ext^1 -graphs of \mathcal{C} and \mathcal{C}' are isomorphic, see examples in 4.5.4. This does not hold for \mathfrak{q}_m : for instance, the half-integral principal block in $\mathcal{F}in(\mathfrak{q}_2)$ and the integral principal blocks in $\mathcal{F}in(\mathfrak{q}_2), \mathcal{F}in(\mathfrak{q}_3)$ have isomorphic ext-graphs and different Ext^1 -graphs (see [27], [21]).

4.8.1.

Take $\mathfrak{g} = \mathfrak{q}_m$. Fix a central character χ and let \mathcal{C}_χ be the corresponding Serre subcategory of $\mathcal{F}in(\mathfrak{q}_m)$. We assume that $\mathcal{C}_\chi \neq 0$ and denote by $(\mathcal{C}_\chi, \text{Ext}^1)$ the Ext^1 -graph of \mathcal{C}_χ .

By above, the set $\text{core}(\lambda)$ is the same for all $L(\lambda) \in \mathcal{C}_\chi$. We denote this set by $\text{core}(\chi)$. We say that χ is Π -invariant if $\text{core}(\chi) \setminus \{0\}$ contains an odd number of elements; in this case each $L(\lambda) \in \mathcal{C}_\chi$ is Π -invariant. If χ is not Π -invariant, then each $L(\lambda) \in \mathcal{C}_\chi$ is not Π -invariant.

We say that χ is integral (resp., half-integral) if $\text{core}(\chi)$ contains an integral (half-integral) number. If $\chi \neq \chi_0$ is atypical, then χ is either integral or half-integral and the graph $(\mathcal{C}_\chi; \text{ext})$ is connected. If $\chi = \chi_0$ and m is even, the graph $(\mathcal{C}_\chi; \text{ext})$ has two connected components $(\mathcal{B}_0; \text{ext})$ and $(\mathcal{B}_{1/2}; \text{ext})$.

If χ is Π -invariant, the graph $(\mathcal{C}_\chi, \text{Ext}^1)$ can be obtained from $(\mathcal{C}_\chi; \text{ext})$ by adding the loops around each vertex λ with $0 \in \{\lambda_i\}_{i=1}^m$, see [21], Theorem 3.1. In particular, $(\mathcal{C}_\chi, \text{Ext}^1) = (\mathcal{C}_\chi; \text{ext})$ if χ is Π -invariant and half-integral.

Consider the case when χ is not Π -invariant. The vertices of $(\mathcal{C}_\chi; \text{Ext}^1)$ are of the forms $(\nu; i)$, where $\nu \in \mathcal{C}_\chi, i \in \mathbb{Z}_2$. By Theorem 3.1 in [21] the graph $(\mathcal{C}_\chi; \text{Ext}^1)$ does not have loops and the vertices $(\nu, i), (\nu, i + 1)$ are connected by a unique edge \longleftrightarrow if $0 \in \{\lambda_i\}_{i=1}^m$; otherwise these vertices are not connected. Consider the edges of the form $(\nu, i) \leftrightarrow (\lambda, j)$. Each edge $\nu \leftrightarrow \lambda$ corresponds to four edges $(\nu, 0) \leftrightarrow (\lambda, j), (\lambda, j) \leftrightarrow (\nu, i)$ and $(\nu, 1) \leftrightarrow (\lambda, j + 1), (\lambda, j + 1) \leftrightarrow (\nu, i + 1)$ for some i, j . By [11],

$$\dim \text{Ext}^1(L(\lambda), L(\nu)) = \dim \text{Ext}^1(L(\nu), \Pi^{\text{tail}(\bar{\lambda}; \bar{\nu})} L(\lambda))$$

which implies $i = 0$ if $\text{tail}(\bar{\lambda}; \bar{\nu})$ is even and $i = 1$ if $\text{tail}(\bar{\lambda}; \bar{\nu})$ is odd. In particular, if χ is not Π -invariant and half-integral, then in the graph $(\mathcal{C}_\chi; \text{Ext}^1)$ the vertices $(\nu, i), (\nu, i+1)$ are not connected and all edges are of the form $\leftarrow \rightarrow$.

Unfortunately, the above information is not sufficient for a description of $(\mathcal{C}_\chi; \text{Ext}^1)$ for atypicality greater than one (the graphs for atypicality one were described in [27], [21]).

Data availability

No data was used for the research described in the article.

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